# ON THE $\mathcal{L}_p$-CONVERGENCE FOR COMPOUND RANDOM SUMS OF PAIRWISE INDEPENDENT RANDOM VARIABLES

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**Abstract -** A random sum is the sum of a random number of random variables, and it has attracted much attention from many researchers. Compound random sums are extensions of classical random sums, in which the random index is determined by the partial sum of independent and identically distributed, non-negative integer-valued random variables. This paper establishes mean convergence theorems for compound random sums of pairwise independent (not necessarily identically distributed) random variables under the Cesàro uniform integrability condition. Our main results extend Theorem 3.1 of Giang and Hung [1] and Theorem 3.1 of Hung [2]. As immediate corollaries, we obtain weak laws of large numbers for negative-binomial sums of pairwise independent random variables.

**Key words -** Compound random sum; negative-binomial sum; Cesàro uniform integrability condition

## 1. Introduction

Limit theorems for random sums are one of the major topics in probability theory. It can be said that the classical work of Robbins [3] laid the foundation for this line of research, where the author established a central limit theorem by replacing the index $n$ in the partial sum with a non-negative integer-valued random variable. Recently, Quang et al. [4] estimated upper bounds on the rates of convergence for random sums of m-orthogonal random variables by using the $K$-functional method.

Let $\{X_j, j \in \mathbb{N}\}$ be a sequence of random variables, and let $Y_1, Y_2, \cdots, Y_n$ be independent and identically distributed (i.i.d.), non-negative integer-valued random variables with a finite mean $\mathbb{E}Y_1 = \alpha$. Then,

$$S_{N_n} = \sum_{j=1}^{N_n} X_j \quad (\text{by convention, } S_0 = 0) \qquad (1)$$

is called a *compound random sum*, where $N_n = \sum_{i=1}^n Y_i$ (see [2] and [5] for more details). Evidently, when $n = 1$, the compound random sum defined by (1) reduces to the classical random sum.

As an example, let $Y_1, Y_2, \cdots, Y_n$ be i.i.d. geometric random variables with parameter $q \in (0,1)$, i.e.,

$$\mathbb{P}(Y_i = k) = q(1-q)^{k-1} \quad (k = 1, 2, \cdots).$$

Then, $N_n = Y_1 + Y_2 + \cdots + Y_n$ follows a negative-binomial distribution with parameters $n \in \mathbb{N}$ and $q \in (0,1)$, that is,

$$\mathbb{P}(N_n = k) = \binom{k-1}{n-1} q^n (1-q)^{k-n}$$

for $k = n, n+1, \cdots$. In this case, the random sum defined in (1) is called a *negative-binomial sum*.

Chen et al. [6] provided bounds in the central limit theorem for compound random sums of i.i.d. random variables using Stein's method. Hung [2] estimated the rates of convergence in weak laws of large numbers for such sums via the Trotter distance. Using the characteristic function method, Giang and Hung [1] established a weak law of large numbers for compound random sums of i.i.d. random variables.

The main purpose of this paper is to extend Theorem 3.1 of Giang and Hung [1], and Theorem 3.1 of Hung [2] to the case of pairwise independent random variables, where the convergence is considered in $\mathcal{L}_p$ for $1 \leq p \leq 2$.

The following lemma provides the von Bahr–Esseen moment inequality for pairwise independent random variables, which plays a crucial role in the proof of the main results.

**Lemma 1.1** (Chen, Bai, and Sung [7]). *Let $1 \leq p \leq 2$, and let $\{X_j, j \in \mathbb{N}\}$ be a sequence of pairwise independent mean zero random variables satisfying $\mathbb{E}|X_j|^p < \infty$ for all $j \in \mathbb{N}$. Then,*

$$\mathbb{E}\left|\sum_{j=1}^n X_j\right|^p \leq C_p \sum_{j=1}^n \mathbb{E}|X_j|^p \quad \text{for each } n \geq 1,$$

*where $C_p$ is a constant depending only on $p$. In the case where $p = 1$ or $p = 2$, we can choose $C_p = 1$.*

The Cesàro uniform integrability condition was introduced by Chandra [9]. A sequence $\{X_n, n \in \mathbb{N}\}$ of random variables is said to be *uniformly integrable in the Cesàro sense* if

$$\lim_{a \to \infty} \sup_{n \geq 1} \left\{ \frac{1}{n} \sum_{j=1}^n \mathbb{E}\left(|X|\mathbf{1}(|X_j| > a)\right) \right\} = 0.$$

Throughout the paper, all random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set of positive integers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, and the set of positive real numbers by $\mathbb{R}_+$. The notation $\xrightarrow{\mathcal{L}_p}$ denotes convergence in $\mathcal{L}_p$, and $\xrightarrow{P}$ stands for convergence in probability.

## 2. Main results

The first main result is stated in the following theorem.

**Theorem 2.1.** *Let $1 \leq p \leq 2$, and let $\{X_j, j \in \mathbb{N}\}$ be a sequence of pairwise independent random variables satisfying*

$$\lim_{a\to\infty} \sup_{k\geq 1} \frac{1}{k}\sum_{j=1}^{k}\mathbb{E}\{|X_j|^p \mathbf{1}(|X_j| > a)\} = 0. \qquad (2)$$

Let $\{Y_i, 1 \leq i \leq n\}$ be a collection of i.i.d., non-negative integer-valued random variables with a finite mean $\mathbb{E}Y_1 = \alpha$. Assume that $\sigma(X_j, j \in \mathbb{N})$ and $\sigma(Y_i, 1 \leq i \leq n)$ are independent for every $n \geq 1$. Then,

$$\frac{1}{n}\sum_{j=1}^{N_n}(X_j - \mathbb{E}X_j) \xrightarrow{\mathcal{L}_p} 0 \quad as \quad n \to \infty,$$

where $N_n = Y_1 + Y_2 + \cdots + Y_n$.

**Proof.** We first note that

$$\mathbb{E}N_n = \mathbb{E}(Y_1 + Y_2 + \cdots + Y_n) = n\alpha.$$

Moreover, by the Lyapunov inequality [8], we have

$$\mathbb{E}(N_n^s) \leq (\mathbb{E}N_n)^s = n^s\alpha^s \text{ for } 0 < s \leq 1.$$

Let $\varepsilon > 0$ be arbitrary. Since (2) holds, there exists a positive constant $M$ such that

$$\sup_{k\geq 1}\frac{1}{k}\sum_{j=1}^{k}\mathbb{E}\{|X_j|^p \mathbf{1}(|X_j| > M)\} < \varepsilon.$$

For each $j \in \mathbb{N}$, set

$$U_j = X_j\mathbf{1}(|X_j| \leq M) \text{ and } V_j = X_j\mathbf{1}(|X_j| > M).$$

By the $c_r$-inequality, we get

$$\mathbb{E}\left|\sum_{j=1}^{N_n}(X_j - \mathbb{E}X_j)\right|^p$$
$$\leq 2^{p-1}\left(\mathbb{E}\left|\sum_{j=1}^{N_n}(U_j - \mathbb{E}U_j)\right|^p + \mathbb{E}\left|\sum_{j=1}^{N_n}(V_j - \mathbb{E}V_j)\right|^p\right).$$

We have

$$\mathbb{E}\left|\sum_{j=1}^{N_n}(U_j - \mathbb{E}U_j)\right|^p$$
$$= \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\mathbb{E}\left|\sum_{j=1}^{k}(U_j - \mathbb{E}U_j)\right|^p\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\left(\mathbb{E}\left|\sum_{j=1}^{k}(U_j - \mathbb{E}U_j)\right|^2\right)^{\frac{p}{2}}\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\left(\sum_{j=1}^{k}\mathbb{E}(U_j - \mathbb{E}U_j)^2\right)^{\frac{p}{2}}\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\left(\sum_{j=1}^{k}\mathbb{E}U_j^2\right)^{\frac{p}{2}}\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)M^p k^{\frac{p}{2}}\right\}$$
$$= M^p\mathbb{E}\left(N_n^{\frac{p}{2}}\right)$$
$$\leq M^p n^{\frac{p}{2}}\alpha^{\frac{p}{2}}.$$

Similarly,

$$\mathbb{E}\left|\sum_{j=1}^{N_n}(V_j - \mathbb{E}V_j)\right|^p$$
$$= \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\mathbb{E}\left|\sum_{j=1}^{k}(V_j - \mathbb{E}V_j)\right|^p\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)C_p\sum_{j=1}^{k}\mathbb{E}|V_j - \mathbb{E}V_j|^p\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)C_p 2^{p-1}\sum_{j=1}^{k}(\mathbb{E}|V_j|^p + \mathbb{E}|\mathbb{E}V_j|^p)\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)C_p 2^p\sum_{j=1}^{k}\mathbb{E}|V_j|^p\right\}$$
$$= \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)C_p 2^p k\left(\frac{1}{k}\sum_{j=1}^{k}\mathbb{E}|V_j|^p\right)\right\}$$
$$\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)C_p 2^p k\varepsilon\right\}$$
$$= C_p 2^p\mathbb{E}(N_n)\varepsilon$$
$$= C_p 2^p n\alpha\varepsilon.$$

Therefore,

$$\mathbb{E}\left|\frac{1}{n}\sum_{j=1}^{N_n}(X_j - \mathbb{E}X_j)\right|^p$$
$$\leq \frac{1}{n^p}2^{p-1}\left(M^p n^{\frac{p}{2}}\alpha^{\frac{p}{2}} + C_p 2^p n\alpha\varepsilon\right)$$
$$= 2^{p-1}\left(M^p n^{-\frac{p}{2}}\alpha^{\frac{p}{2}} + C_p 2^p\alpha\varepsilon n^{1-p}\right).$$

If $1 < p \leq 2$, then it is obvious that

$$\mathbb{E}\left|\frac{1}{n}\sum_{j=1}^{N_n}(X_j - \mathbb{E}X_j)\right|^p \to 0 \text{ as } n \to \infty.$$

In the case where $p = 1$, since $\varepsilon > 0$ is arbitrary, by letting $\varepsilon \to 0$, it follows that

$$\mathbb{E}\left|\frac{1}{n}\sum_{j=1}^{N_n}(X_j - \mathbb{E}X_j)\right| \to 0 \text{ as } n \to \infty.$$

The proof is finished.

**Remark 2.1.** Assumption (2) means that $\{|X_j|^p, j \in \mathbb{N}\}$ is *uniformly integrable in the Cesàro sense* for $1 \leq p \leq 2$. Furthermore, (2) holds when $\{X_j, j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with finite mean $\mathbb{E}X_1 = \mu$. Hence, Theorem 2.1 yields Theorem 3.1 of Giang and Hung [1].

The following corollary establishes mean convergence for negative-binomial sums of pairwise independent random variables.

**Corollary 2.1.** Let $1 \leq p \leq 2$, and let $\{X_j, j \in \mathbb{N}\}$ be a sequence of pairwise independent random variables satisfying

$$\lim_{a \to \infty} \sup_{k \geq 1} \frac{1}{k} \sum_{j=1}^{k} \mathbb{E}\{|X_j|^p \mathbf{1}(|X_j| > a)\} = 0.$$

Let $N_{n,q}$ be a negative-binomial random variable with parameters $n \in \mathbb{N}$ and $q \in (0,1)$, and assume that $\sigma(X_j, j \in \mathbb{N})$ and $\sigma(N_{n,q})$ are independent for all $n \geq 1$, $q \in (0,1)$. Then,

$$\frac{1}{n} \sum_{j=1}^{N_{n,q}} (X_j - \mathbb{E}X_j) \xrightarrow{\mathcal{L}_p} 0 \quad as \quad n \to \infty.$$

**Proof.** Let $Y$ be a geometrically distributed random variable with parameter $q \in (0,1)$, and denote by

$$f_Y(t) = \frac{qe^{it}}{1 - (1-q)e^{it}} \quad (t \in \mathbb{R})$$

the characteristic function of $Y$. Let $Y_1, Y_2, \cdots, Y_n$ be independent copies of $Y$. Then, the characteristic function of $Y_1 + Y_2 + \cdots + Y_n$ will be

$$\begin{aligned}
f_{Y_1+Y_2+\cdots+Y_n}(t) &= \mathbb{E}\{e^{it(Y_1+Y_2+\cdots+Y_n)}\} \\
&= \mathbb{E}(e^{itY_1})\mathbb{E}(e^{itY_2}) \cdots \mathbb{E}(e^{itY_n}) \\
&= \{f_Y(t)\}^n \\
&= \left\{\frac{qe^{it}}{1 - (1-q)e^{it}}\right\}^n \\
&= f_{N_{n,q}}(t),
\end{aligned}$$

where $f_{N_{n,q}}(t)$ denotes the characteristic function of $N_{n,q}$. This implies that

$$N_{n,q} =^d Y_1 + Y_2 + \cdots + Y_n,$$

and hence, Corollary 2.1 follows from Theorem 2.1.

The following theorem is a variant of Theorem 2.1. This result extends Theorem 3.1 of Hung [2].

**Theorem 2.2.** Let $1 \leq p \leq 2$, and let $\{X_j, j \in \mathbb{N}\}$ be a sequence of pairwise independent random variables such that $\{|X_j|^p, j \in \mathbb{N}\}$ is uniformly integrable in the Cesàro sense. Let $\{Y_i, 1 \leq i \leq n\}$ be a collection of i.i.d., non-negative integer-valued random variables with a finite mean $\mathbb{E}Y_1 = \alpha$. Assume that $\sigma(X_j, j \in \mathbb{N})$ and $\sigma(Y_i, 1 \leq i \leq n)$ are independent for all $n \in \mathbb{N}$. If

$$\lim_{n \to \infty} \mathbb{E}(N_n^{-\gamma}) = 0 \quad for \ all \quad \gamma \in (0,1], \tag{3}$$

then,

$$\frac{1}{N_n} \sum_{j=1}^{N_n} (X_j - \mathbb{E}X_j) \xrightarrow{\mathcal{L}_p} 0 \quad as \quad n \to \infty,$$

where $N_n = Y_1 + Y_2 + \cdots + Y_n$.

**Proof.** Let $\varepsilon > 0$ be arbitrary. Since $\{|X_j|^p, j \in \mathbb{N}\}$ is uniformly integrable in the Cesàro sense, there exists a positive constant M such that

$$\sup_{k \geq 1} \frac{1}{k} \sum_{j=1}^{k} \mathbb{E}\{|X_j|^p \mathbf{1}(|X_j| > M)\} < \varepsilon.$$

For each $j \in \mathbb{N}$, set

$$U_j = X_j \mathbf{1}(|X_j| \leq M) \quad and \quad V_j = X_j \mathbf{1}(|X_j| > M).$$

By the $c_r$-inequality, we get

$$\begin{aligned}
&\mathbb{E}\left|\frac{1}{N_n} \sum_{j=1}^{N_n} (X_j - \mathbb{E}X_j)\right|^p \\
&\leq 2^{p-1}\left(\mathbb{E}\left|\frac{1}{N_n}\sum_{j=1}^{N_n}(U_j - \mathbb{E}U_j)\right|^p \right. \\
&\qquad\qquad \left. + \mathbb{E}\left|\frac{1}{N_n}\sum_{j=1}^{N_n}(V_j - \mathbb{E}V_j)\right|^p\right).
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}\left|\frac{1}{N_n}\sum_{j=1}^{N_n}(U_j - \mathbb{E}U_j)\right|^p \\
&= \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\mathbb{E}\left|\frac{1}{k}\sum_{j=1}^{k}(U_j - \mathbb{E}U_j)\right|^p\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\frac{1}{k^p}\left(\mathbb{E}\left|\sum_{j=1}^{k}(U_j - \mathbb{E}U_j)\right|^2\right)^{\frac{p}{2}}\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\frac{1}{k^p}\left(\sum_{j=1}^{k}\mathbb{E}(U_j - \mathbb{E}U_j)^2\right)^{\frac{p}{2}}\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\frac{1}{k^p}\left(\sum_{j=1}^{k}\mathbb{E}U_j^2\right)^{\frac{p}{2}}\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)k^{-\frac{p}{2}}M^p\right\} \\
&= M^p \, \mathbb{E}\left(N_n^{-\frac{p}{2}}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}\left|\frac{1}{N_n}\sum_{j=1}^{N_n}(V_j - \mathbb{E}V_j)\right|^p \\
&= \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\mathbb{E}\left|\frac{1}{k}\sum_{j=1}^{k}(V_j - \mathbb{E}V_j)\right|^p\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\frac{1}{k^p}C_p\sum_{j=1}^{k}\mathbb{E}|V_j - \mathbb{E}V_j|^p\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\frac{C_p}{k^p}2^{p-1}\sum_{j=1}^{k}(\mathbb{E}|V_j|^p + \mathbb{E}|\mathbb{E}V_j|^p)\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)\frac{C_p}{k^p}2^p\sum_{j=1}^{k}\mathbb{E}|V_j|^p\right\} \\
&\leq \sum_{k=1}^{\infty}\left\{\mathbb{P}(N_n = k)k^{1-p}C_p2^p\left(\frac{1}{k}\sum_{j=1}^{k}\mathbb{E}|V_j|^p\right)\right\}
\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \left\{ \mathbb{P}(N_n = k) k^{1-p} C_p 2^p \varepsilon \right\}$$

$$= \mathbb{E}(N_n^{1-p}) C_p 2^p \varepsilon.$$

Therefore,

$$\mathbb{E}\left| \frac{1}{N_n} \sum_{j=1}^{N_n} (X_j - \mathbb{E}X_j) \right|^p$$

$$\leq 2^{p-1} \left\{ M^p \, \mathbb{E}\left( N_n^{-\frac{p}{2}} \right) + \mathbb{E}(N_n^{1-p}) C_p 2^p \varepsilon \right\}.$$

Under assumption (3), if $1 < p \leq 2$, then

$$\mathbb{E}\left| \frac{1}{N_n} \sum_{j=1}^{N_n} (X_j - \mathbb{E}X_j) \right|^p \to 0 \quad \text{as} \quad n \to \infty.$$

In the case where $p = 1$, since $\varepsilon > 0$ is arbitrary, by letting $\varepsilon \to 0$, it follows that

$$\mathbb{E}\left| \frac{1}{n} \sum_{j=1}^{N_n} (X_j - \mathbb{E}X_j) \right| \to 0 \quad \text{as} \quad n \to \infty.$$

The proof is finished.

The following corollary is a variant of Corollary 2.1.

**Corollary 2.2.** *Let* $1 \leq p \leq 2$, *and let* $\{X_j, j \in \mathbb{N}\}$ *be a sequence of pairwise independent random variables such that* $\{|X_j|^p, j \in \mathbb{N}\}$ *is uniformly integrable in the Cesàro sense. Let* $N_{n,q}$ *be a negative-binomial random variable with parameters* $n \in \mathbb{N}$ *and* $q \in (0,1)$, *and assume that* $\sigma(X_j, j \in \mathbb{N})$ *and* $\sigma(N_{n,q})$ *are independent for all* $n \geq 1$, $q \in (0,1)$. *Then,*

$$\frac{1}{N_{n,q}} \sum_{j=1}^{N_{n,q}} (X_j - \mathbb{E}X_j) \xrightarrow{\mathcal{L}_p} 0 \quad as \quad n \to \infty.$$

**Proof.** We first observe that

$$\frac{1}{a_1 + a_2 + \cdots + a_n} \leq n^{-2} \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \quad (4)$$

for all $a_i > 0$ $(1 \leq i \leq n)$.

Let $Y_1, Y_2, \cdots, Y_n$ be i.i.d. geometric random variables with parameter $q \in (0,1)$. Then,

$$\mathbb{E}(Y_1^{-1}) = \sum_{k=1}^{\infty} \frac{1}{k} q(1-q)^{k-1}$$

$$= \frac{q}{1-q} \sum_{k=1}^{\infty} \frac{(1-q)^k}{k} = \frac{q}{1-q} \ln \frac{1}{q}.$$

Hence, by (4), we obtain

$$\mathbb{E}(N_{n,q}^{-1}) \leq n^{-1} \mathbb{E}(Y_1^{-1}) \leq n^{-1} \frac{q}{1-q} \ln \frac{1}{q}.$$

In the case where $0 < \gamma < 1$, applying (4) again, we also have

$$\mathbb{E}(N_{n,q}^{-\gamma}) = \mathbb{E}\left( \frac{1}{Y_1 + Y_2 + \cdots + Y_n} \right)^{\gamma}$$

$$\leq \mathbb{E}\left\{ n^{-2\gamma} \left( \frac{1}{Y_1} + \frac{1}{Y_2} + \cdots + \frac{1}{Y_n} \right)^{\gamma} \right\}$$

$$\leq n^{-2\gamma-1} \mathbb{E}(Y_1^{-\gamma})$$

$$\leq n^{-2\gamma-1} \frac{q^{\gamma}}{1-q} \Gamma(1-\gamma) \quad \text{(Proposition 3.1 in [10])}.$$

Since $\Gamma(1-\gamma) = \int_0^{\infty} t^{-\gamma} e^{-t} dt < \infty$ for $\gamma \in (0,1)$, it follows that

$$n^{-2\gamma-1} \frac{q^{\gamma}}{1-q} \Gamma(1-\gamma) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\lim_{n \to \infty} \mathbb{E}(N_{n,q}^{-\gamma}) = 0 \quad \text{for all} \quad \gamma \in (0,1],$$

that is, the negative-binomial random variable $N_{n,q}$ satisfies assumption (3) of Theorem 2.2. Hence, Corollary 2.2 follows directly from Theorem 2.2. The proof is finished.

## 3. Conclusion

Our main results are based on the moment inequalities established by Chen, Bai, and Sung [7]. Therefore, for dependent structures, these results can be extended provided that suitable moment inequalities are available.

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