A STUDY OF UNSTEADY HEAT TRANSFER IN A FIN SUBJECTED TO PERIODIC BOUNDARY CONDITIONS

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Abstract - A numerical model was developed based on the finite difference method (FDM) to solve the transient heat transfer problem through a fin with periodic oscillating boundary conditions. The model was validated against a previous study and demonstrated excellent agreement. It was applied to investigate the effect of thermal oscillating boundaries on the temperature distribution and thermal efficiency of the fin under unsteady conditions. The results showed that the temperature distribution and thermal efficiency nearly reached a steady oscillation after just the first cycle for small values of parameter M, and the difference between the first and subsequent cycles gradually decreased as M increased. The model developed in this study can be applied to research on practical fin applications with nonlinear thermal boundary oscillations, such as solar energy collectors, heat sinks in electronic devices, and internal combustion engines.

Key words - Fins; periodic boundary; rectangular fin; fin efficiency; nonlinear heat transfer

1. Introduction

In electronic components, internal combustion engines, solar energy collectors, etc., fins are commonly used to enhance heat transfer. In these engineering applications, fins often operate under thermal oscillation conditions. Thermal oscillations can originate from within the device itself or from fluctuations in the surrounding environment. Many studies have been conducted on the effect of thermal oscillations on the performance of fins. Various types of periodic boundary conditions have been considered, such as oscillating base temperature, oscillating heat flux, oscillating ambient temperature, or combinations of these conditions. More detailed information on periodic heat transfer in fins can be found in the review by Aziz and Lunardini [1].

Many studies on fins with thermally oscillating boundaries have used analytical methods [2–6]. However, most of these studies simplify the heat transfer problem by assuming it to be linear. This assumption makes it easier to solve periodic problems using analytical approaches. In reality, many physical phenomena in fins are nonlinear because the heat transfer coefficient and thermal conductivity are temperature-dependent, or due to radiative heat transfer occurring between the fin surfaces and the environment. These nonlinear characteristics present significant challenges for analytical methods, and in many cases, exact solutions are not possible. Aziz and Na [7] applied perturbation analysis, a semi-numerical approach, to solve periodic heat transfer in radiating fins. The zero-order problem, corresponding to steady-state fin behavior, was solved using quasi linearization. A complex combination method, together with a non-iterative numerical scheme, was employed to solve the first- and second-order problems. The same method was also applied to periodic heat transfer in fins with temperature-dependent thermal conductivity and coordinate-dependent heat transfer coefficients [8]. Yang et al. [9] used the double decomposition method to solve periodic heat transfer in a convective rectangular fin with an oscillating base temperature and temperature-dependent thermal conductivity. Chiu and Chen [10] applied Adomian's decomposition method to solve nonlinear heat transfer in an annular convective fin subjected to a periodic boundary condition.

The application of numerical methods to study periodic heat transfer in fins has been reported in several studies. Al-Sanea and Mujahid [11] used the finite volume method (FVM) with a fully implicit formulation to investigate periodic heat transfer in a longitudinal rectangular fin with constant thermal properties and uniform convective heat transfer along the fin. Their method showed excellent agreement with the exact solution and was used to study temperature distribution, heat flow rate, and thermal performance under the influence of various parameters. Eslinger and Chung [12] studied heat transfer from a convective-radiative fin array using the finite element method (FEM), taking into account fin-to-fin, fin-to-base, and fin-to-environment radiative interactions. The fin base temperature varies periodically, and thermal properties are assumed to be constant. Campo [13] employed the FDM to study a convective-radiative circular fin with a periodic temperature boundary at the base. Both temperature distribution and heat flux were investigated. Other numerical methods, such as the spectral finite volume method, the lattice Boltzmann method, and hybrid numerical techniques, have also been applied to study periodic heat transfer under Fourier and non-Fourier effects, as reported in studies [14–16]. To the best of the author's knowledge, most studies on periodic heat transfer in fins under Fourier's law focus on steady-state conditions.

In this study, a numerical model based on the finite difference method is developed to investigate nonlinear heat transfer through a straight fin under a periodic temperature boundary condition. The model is applied to study the temperature distribution and thermal efficiency under unsteady conditions. The effects of other parameters on the thermal efficiency of the fin are also investigated.

2. Mathematical formulation

2.1. Problem statement

Consider a straight fin with a constant rectangular cross-sectional area A_c which exchanges heat with the surrounding environment through convection and radiation, as shown in Figure 1. The fin has a length L, width W, and thickness δ_b. The base temperature T_b oscillates periodically, and all surfaces of the fin exposed to the environment are in contact with a uniform ambient temperature T_a. The initial temperature of the fin is equal to the ambient temperature. The amount of heat exchanged at the fin tip is negligible, and the tip is assumed to be adiabatic.

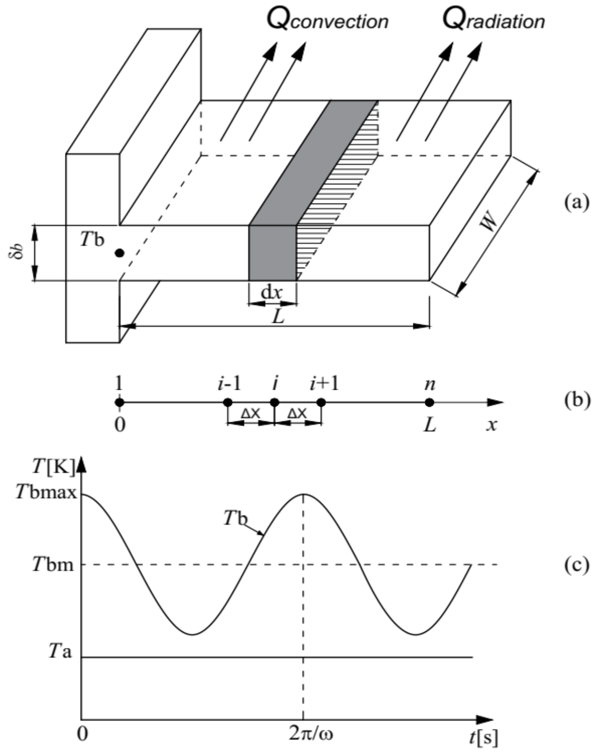

Figure 1. *Rectangular straight fin a) fin geometry, b) FDM mesh, and (c) periodic boundary condition*

The differential heat transfer equation describing the heat transfer through the fin is derived from the heat balance equation for a fin element of length dx is given as:

$$\rho c_\mathrm{p} A_\mathrm{c} \frac{\partial T(x,t)}{\partial t} = \lambda A_\mathrm{c} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right) - \frac{\alpha P}{A_\mathrm{c}}(T - T_\mathrm{a}) - \varepsilon \sigma P (T^4 - T_\mathrm{a}^4) \quad (1)$$

where ρ is the density (kg/m³); c_p is the specific heat capacity (Jkg^{-1}K^{-1}); t is time (s); T is the fin temperature (K); λ is the heat conductivity of the fin (Wm^{-1}K^{-1}); α is the convective heat transfer coefficient (Wm^{-2}K^{-1}); ε is the emissivity of the fin surface; $\sigma = 5.67 \times 10^{-8}$(Wm^{-2}K^{-4}) is the Stefan-Boltzmann constant; P is the perimeter of the fin cross-section.

The base temperature of the fin oscillates periodically with a dimensionless amplitude $A = (T_{bmax} - T_{bm})/(T_{bm} - T_a) < 1$ and angular frequency ω, as specified by the boundary condition in [2]:

$$x = 0,\ T_b = T_{bm} + (T_{bm} - T_a) A\cos(\omega t) \quad (2)$$

The boundary condition describing the adiabatic tip of the fin is as follows:

$$x = L,\ \frac{dT}{dx} = 0 \quad (3)$$

By using the dimensionless quantities:

$$X = \frac{x}{L}, \theta = \frac{T}{T_\mathrm{bm}}, \theta_\mathrm{a} = \frac{T_\mathrm{a}}{T_\mathrm{bm}}, \tau = \frac{\lambda t}{\rho c_\mathrm{p} L^2},$$
$$B = \frac{\omega \rho c_\mathrm{p} L^2}{\lambda}, M^2 = \frac{\alpha P L^2}{\lambda A_\mathrm{c}},\quad N_\mathrm{R} = \frac{\varepsilon \sigma P L^2 T_\mathrm{bm}^3}{\lambda A_\mathrm{c}} \quad (4)$$

The differential equation (1) can be rewritten as:

$$\frac{\partial \theta}{\partial \tau} = \frac{d^2\theta}{dX^2} - M^2(\theta - \theta_\mathrm{a}) - N_R(\theta^4 - \theta_\mathrm{a}^4) \quad (5)$$
$$0 \le X \le 1$$

And the boundary conditions are rewritten as follows:

$$X = 0:\quad \theta(0,\tau) = 1 + (1 - \theta_\mathrm{a}) A\cos(B\tau)$$
$$X = 1:\ \frac{d\theta}{dX} = 0 \quad (6)$$

2.2. Numerical method

The finite difference method is applied to solve Equation (5) with the boundary conditions given in Equation (6). According to this method, the fin is divided into *n* nodes, numbered from 1 to n corresponding to the temperatures of nodes T_1 to T_n. The second-order derivative term in the equation for an interior node i ($i = 2 \div n - 1$) is approximated using central difference as follows:

$$\left.\frac{d^2\theta}{dX^2}\right|_i \cong \frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{\Delta X^2} \quad (7)$$

+ Approximation of the time-dependent term: the implicit scheme is applied.

$$\frac{\partial \theta_i^{\{k+1\}}}{\partial \tau} = \frac{\theta_i^{\{k+1\}} - \theta_i^{\{k\}}}{\Delta \tau} \quad (8)$$

+ Interior nodes

For the interior nodes $i = 2 \div (n-1)$, substituting Equations (7) and (8) into Equation (5), we obtain:

$$\frac{\theta_i^{\{k+1\}} - \theta_i^{\{k\}}}{\Delta \tau} = \frac{\theta_{i-1}^{\{k+1\}} - 2\theta_i^{\{k+1\}} + \theta_{i+1}^{\{k+1\}}}{\Delta X^2} - M^2\left(\theta_i^{\{k+1\}} - \theta_\mathrm{a}\right) - N_\mathrm{R}\left(\left(\theta_i^{\{k+1\}}\right)^4 - \theta_\mathrm{a}^4\right) \quad (9)$$

+ Boundary nodes

- Nodes *1:*

$$\theta_1 = 1 + (1 - \theta_\mathrm{a}) A\cos(B\tau) \quad (10)$$

- Nodes *n*, apply the balance equation we have:

$$\frac{\theta_n^{\{k+1\}} - \theta_n^{\{k\}}}{\Delta \tau} = 2\left(\frac{\theta_{n-1}^{\{k+1\}} - \theta_n^{\{k+1\}}}{\Delta X^2}\right) - M^2\left(\theta_n^{\{k+1\}} - \theta_\mathrm{a}\right) - N_R\left(\left(\theta_n^{\{k+1\}}\right)^4 - \theta_\mathrm{a}^4\right) \quad (11)$$

As a result of applying the finite difference method to Equation (5), a system of n nonlinear algebraic equations is obtained by combining Equations (9), (10), and (11). The open-source programming language Python is used to solve this system with a convergence tolerance of 10^{-7}.

Different grid densities are used to verify the independence of the results from the mesh resolution for various values of M in the range 0 to 5. Higher value of M can lead to significantly lower average heat transfer efficiency. The results indicate that the temperature distribution in the fin changes insignificantly when the number of nodes exceeds 25. Therefore, in this study, the number of grid elements is set to $n = 30$.

3. Fin efficiency and model validation

Fin efficiency is defined as the ratio between the actual amount of heat exchanged between the fin surface and the environment ($\dot{Q}_{\mathrm{f}}$) and the maximum possible heat exchanged between the fin and the environment ($\dot{Q}_{\mathrm{max}}$). For fins with oscillating thermal boundaries, the commonly used measures are instantaneous efficiency and average efficiency. The instantaneous efficiency at time step $\{k\}$ is defined as:

$$\eta(\tau) = \frac{\dot{Q}_{\mathrm{f}}^{\{k\}}}{\dot{Q}_{\mathrm{max}}^{\{k\}}} \tag{12}$$

- The average efficiency for a period of the oscillation cycle of the fin can be determined as follows:

$$\eta_{\mathrm{ave}} = \frac{1}{\tau_0}\int_0^{\tau_0} \eta(\tau)d\tau \tag{13}$$

where τ_0 is the duration of one periodic oscillation cycle, defined as:

$$\tau_0 = 2\pi/\omega$$

For validation, the numerical model developed in this study is compared with the analytical method, also known as the exact method, developed by Yang [2]. The temperature distribution along the fin is determined for different values of $B\tau = 0.2\pi$; $\pi/2$; $3\pi/2$; π with M=1.0, $B = 1.0$, $N_{\mathrm{R}} = 0$ và $\theta_{\mathrm{a}} = 0.6$. Due to the difference in the dimensionless temperature used in this study, θ, and that in Yang's study [2], θ_{ref}, a conversion is performed for comparison. The relationship between the two temperatures is expressed as follows:

$$\theta_{\mathrm{ref}} = \frac{\theta - \theta_{\mathrm{a}}}{1 - \theta_{\mathrm{a}}} \tag{14}$$

4. Results and discussion

Figure 2 shows the comparison of the temperature distribution along the fin between the finite difference method and the analytical method for different values of $B\tau$. The results indicate that the model developed in this study has very high accuracy, with the relative error in all cases being less than 0,008%.

Figure 3 shows the temperature oscillations over time at the points X =0; 0.5; 1.0 along the fin, with the parameters A=0.5, $M = 1.0$, $N_R = 0.5$ và $\theta_{\mathrm{a}} = 0.6$. The results indicate that the amplitude of oscillation decreases from the base to the tip of the fin. In addition to the temperature distribution at the base of the fin matching the boundary condition, the temperature oscillation amplitudes at $X = 0.5$ and $X = 1.0$ stabilize almost immediately after the first cycle. The results also show a phase shift in the oscillations along the fin, with the delay increasing from the base to the tip.

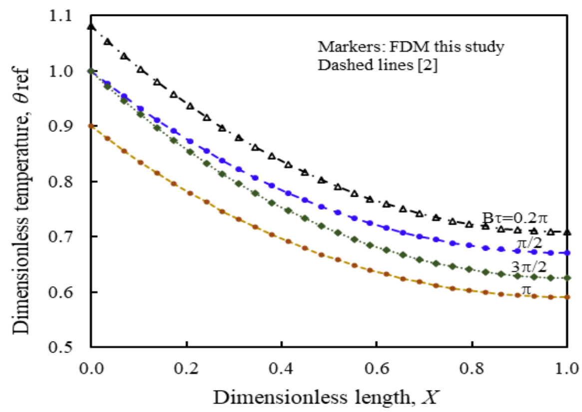

Figure 2. *Comparison of dimensionless temperature distributions from FDM and exact solution with M =1.0,* $B = 1.0$, $N_R = 0$, $A = 0.1$

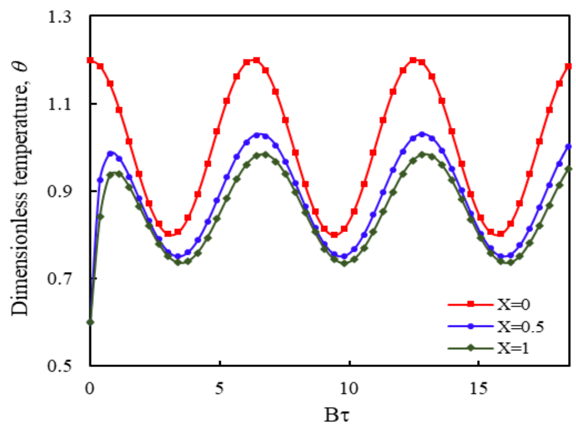

Figure 3. *Temperature along the fin at* $X = 0; 0.5; 1$ *with A=0.5,* $M = 1.0$, $N_R = 0.5$ *and* $\theta_a = 0.6$

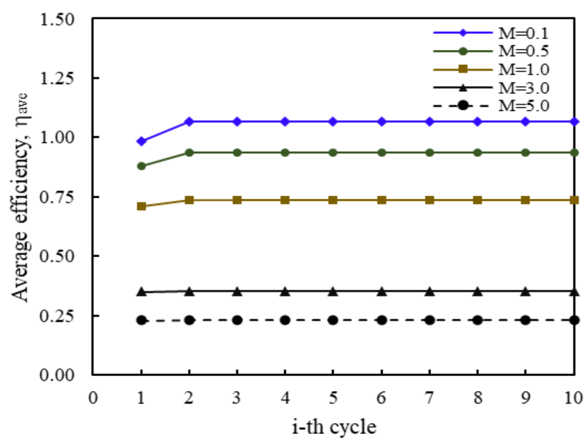

Figure 4. *Average efficiency at the i-th cycle for various M, with A=0.5,* $B = 1.0$, *NR=0.5,* $\theta_a = 0.6$

The effect of the parameter M on the average thermal efficiency is shown in Figure 4. The results indicate that the thermal efficiency decreases as M increases. This can be explained by the fact that as M increases, it corresponds to an increase in the heat transfer coefficient α or a decrease in the heat conductivity coefficient λ, which leads to a decrease in the temperature distribution within the fin. This can be clearly seen from the temperature distribution $X = 0.5$ for different values of M. The decrease in temperature within the fin leads to a reduction in the amount of heat dissipated from the fin surface to the environment, causing the fin efficiency to drop. From Figure 4, it can be observed that after the first cycle, the average thermal efficiency of subsequent cycles remains almost unchanged over time. This suggests that the fin reaches a stable oscillation. This phenomenon is similar to the change in temperature distribution along the fin, as mentioned in Figures 3 and 5. The degree of change in efficiency between the second and first cycles also decreases as M increases, with $M = 3; 5$ showing almost no difference in average efficiency between the first and second cycles.

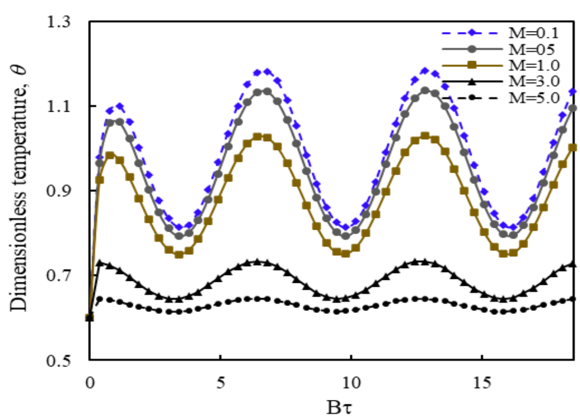

Figure 5. *Temperature distribution along the fin for varying M with A=0.5, B=1,* $N_R = 0.5$, $\theta_a = 0.6$

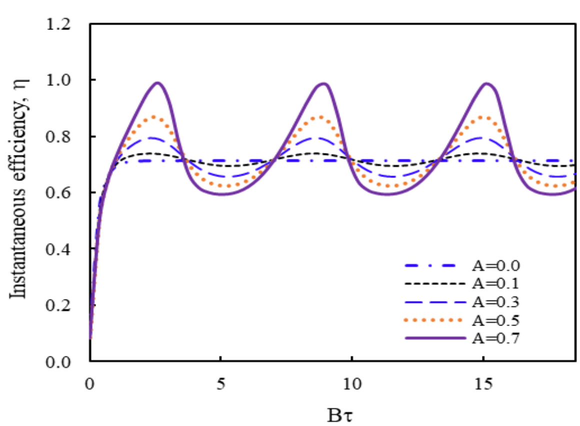

Figure 6. *Dependence of instantaneous efficiency on A with* $M = 1.0$, $N_R = 0.5$, $B = 1.0$, $\theta_a = 0.6$

Figure 6 illustrates the effect of the oscillation amplitude A on the instantaneous thermal efficiency during the first three oscillation cycles. The results show that the amplitude of the thermal efficiency oscillation increases as A increases. The results also indicate a phase shift in the oscillation of efficiency. As A increases, the average thermal efficiency also increases. The results from Table 1 show that the average thermal efficiency increases by approximately 6.4% for $A = 0.9$ compared to the case with no thermal oscillation ($A = 0$).

Table 1. *Time-averaged fin efficiency for varying A, with* $M = 1.0$, $B = 1.0$, $N_R = 0.5$, $\theta_a = 0.6$

Number of cycles	A				
	0.0	0.1	0.5	0.7	0.9
1	0.688	0.688	0.696	0.709	0.752
5	0.710	0.710	0.718	0.731	0.774
10	0.712	0.712	0.721	0.733	0.777
20	0.714	0.714	0.722	0.735	0.778
30	0.714	0.714	0.722	0.735	0.778

Figure 7 shows the instantaneous thermal efficiency for different values of the oscillation frequency B over a specific time period corresponding to the duration of one oscillation cycle with B=0.1. The results show that as B increases, the amplitude of the instantaneous thermal efficiency oscillation increases significantly, and the phase shift becomes more pronounced. Table 2 illustrates the change in average thermal efficiency for different values of B over various time intervals. The results show that the average thermal efficiency remains almost constant over time for a given value of B. However, as B increases, the average thermal efficiency also increases, with a rise of approximately 7.7% within the range of B values considered under steady-state conditions.

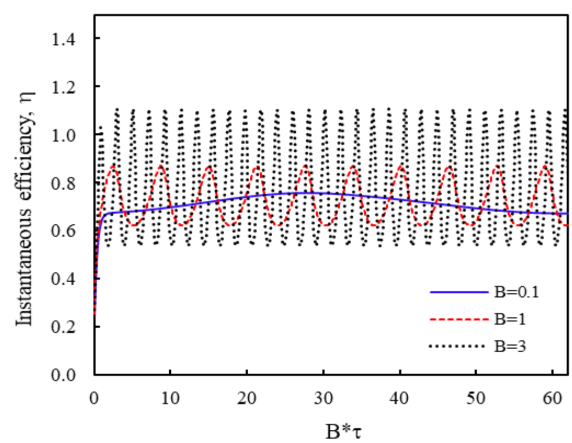

Figure 7. *Dependence of instantaneous efficiency on B, with* $M = 1.0$, $N_R = 0,5$, *A=0.5*, $\theta_a = 0.6$

Table 2. *Time-averaged fin efficiency for varying B, with* $M = 1.0$, $N_R = 0.5$, *A=0.5*, $\theta_a = 0.6$

τ	B			
	0.1	1	3	5
20π	0.712	0.721	0.760	0.789
40π	0.713	0.722	0.762	0.790
60π	0.714	0.723	0.762	0.791
80π	0.714	0.723	9.763	0.791
100π	0.714	0.723	0.763	0.791

5. Conclusion

The study has developed a numerical model to solve the nonlinear heat transfer problem through a straight fin with a rectangular cross-section and periodic oscillating boundaries based on the FDM. The Python programming language was used to solve the nonlinear algebraic equation system. The model results were compared with the exact solutions from previous studies, showing that the model provides very high accuracy.

The model was applied to investigate the temperature distribution and thermal efficiency of the fin under unsteady conditions. The results indicate that the temperature distribution and thermal efficiency almost reach a steady state after the first cycle. Additionally, the difference between the first cycle and subsequent cycles decreases as dimensionless parameter M increases.

The model was also applied to investigate the effect of amplitude (A) and oscillation frequency (B) on instantaneous thermal efficiency and average thermal efficiency. The results show that as A and B increase, the amplitude of oscillation of the temperature distribution and thermal efficiency increases, and the average thermal efficiency also increases.

The model can be applied to study the use of fins in practical applications with thermally oscillating boundary conditions, such as solar collectors, heat sinks in electronic devices, internal combustion engines.

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