CONVERGENCE OF WEIGHTED SUMS OF PAIRWISE INDEPENDENT STOCHASTICALLY DOMINATED RANDOM VARIABLES

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Abstract - Limit theorems play an important role in probability theory and have numerous applications in statistics. Among them, the weak and strong laws of large numbers established by Kolmogorov and Marcinkiewicz–Zygmund for sequences of independent and identically distributed random variables are classical results of fundamental significance. These results have attracted considerable attention and have been extended in various directions under more general settings. In this paper, we further develop these classical results for sequences of pairwise independent random variables. The assumption of identical distribution is replaced by a weaker condition, in which the random variables are stochastically dominated by another random variable with a heavy-tailed distribution. Based on this setting, we establish limit theorems for weighted sums of random variables, thereby extending the applicability of the laws of large numbers to more general conditions. By applying these results, we investigate the convergence for a nonparametric regression estimator.

Key words - Weighted sums; pairwise independence; stochastically dominated; laws of large numbers; nonparametric regression

1. Introduction

The Kolmogorov and Marcinkiewicz–Zygmund laws of large numbers, originally formulated for sequences of independent and identically distributed random variables, have attracted considerable attention and have been extended by many authors. Sung [1] extended these results to sequences of pairwise independent and identically distributed random variables. Dung [2] established laws of large numbers for sequences of weighted sums of independent random variables with infinite moments. The works of Xuan [3] and Chau [4] further investigated laws of large numbers for pairwise independent random variables with heavy-tailed distributions. More recently, Tu [5] also established laws of large numbers for sequences of pairwise independent randomly bounded random variables under finite moment conditions.

Let $\{X_n; n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ and let $\{a_{ni}; 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers. In many applications of statistics, one is interested in weighted sums of the following form:

$$S_n = \sum_{i=1}^{n} a_{ni} X_i.$$

Therefore, the study of the convergence of sequences of weighted sums of random variables is of great significance in practical applications. In this paper, we establish the weak law of large numbers and the strong law of large numbers for sequences of weighted sums of pairwise independent random variables that are stochastically dominated by a random variable with a heavy-tailed distribution. Our results extend those of Xuan et al. [3] and are applied to the convergence analysis of a nonparametric regression estimator.

For two sequences of positive real numbers $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$, we use the notation $a_n \asymp b_n$ to indicate that $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$. The notation $a_n = o(b_n)$ means that $\lim\limits_{n\to\infty} a_n/b_n = 0$ and $a_n \sim b_n$ is used when $\lim\limits_{n\to\infty} a_n/b_n = 1$. These notations are also applied to positive real-valued functions $f(x)$ and $g(x)$. The indicator function of a set A is denoted by $I(A)$. Throughout the paper, the symbol C denotes a generic positive constant $(0 < C < \infty)$, whose value may vary from one occurrence to another.

Definition 1. *A sequence of random variables* $\{X_n, n \geq 1\}$ *is called pairwise independent if any two random variables* $X_i, X_j, i \neq j$, *are independent.*

Definition 2. *A sequence of random variables* $\{X_n, n \geq 1\}$ *is said to be stochastically dominated by a random variable* X *if for every* $t > 0$ *one has:*

$$\sup_n P(|X_n| > t) \leq P(|X| > t).$$

Definition 3. *Let* $a \geq 0$. *A positive measurable function* $f(x)$ *defined on the interval* $[a; \infty)$ *is called slowly varying at infinity if*

$$\frac{f(tx)}{f(t)} \to 1 \text{ as } t \to \infty \text{ for every } x > 0.$$

For $x > 0$, we denote $\log^+(x) = \max\{1, \ln(x)\}$, where $\ln(x)$ is the natural logarithm. Clearly, the functions $\log^+(x)$ and $\log^+(\log^+(x))$ are slowly varying at infinity. Furthermore, the product of two slowly varying functions is also slowly varying.

Definition 4. *A sequence of random variables* $\{X_n, n \geq 1\}$ *is said to converge in probability to a random variable* X *if for every* $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0.$$

We denote this by $X_n \xrightarrow{P} X$ *as* $n \to \infty$.

A sequence of random variables $\{X_n, n \geq 1\}$ *is said to converge almost surely to a random variable* X *if*

$$P\left(\omega: \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1.$$

We denote this by $X_n \to X\ (a.s.)$ *as* $n \to \infty$.

In the proofs of the main results of this paper, we shall make use of the following lemmas:

Lemma 1. [1] *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables and define* $S_n = \sum_{i=1}^{n} X_i$. *Let* $\{b_n, n \geq 1\}$ *be a sequence of positive numbers increasing to infinity such that*

$$0 < b_n \uparrow +\infty \ \text{ and } \ \frac{b_{2n}}{b_n} = O(1).$$

If

$$\sum_{n=1}^{+\infty} \frac{1}{n} P\left(\max_{1\leq i\leq n} |S_i| > \varepsilon b_n\right) < +\infty,$$

for every $\varepsilon > 0$, *then* $S_n/b_n \to 0$ *(a.s.).*

Lemma 2. [5] *Let* $p, q > 0$. *If* $\{X_n, n \geq 1\}$ *is a sequence of random variables that is stochastically dominated by a random variable* X *with* $E(|X|^p) < +\infty$, *then for every* $n \geq 1$ *and* $x > 0$, *the following inequalities hold:*

i) $E(|X_n|^q I\{|X_n| \leq x\})$
$\leq E(|X|^q I\{|X| \leq x\}) + x^q P(|X| > x)$.

ii) $E(|X_n|^q I\{|X_n| > x\}) \leq E(|X|^q I\{|X| > x\})$ *under the assumption that* $q \leq p$.

Lemma 3. [2] *Let* $1 < r < 2$ *and let* X *be a random variable. Suppose that* $P(|X| > x) \asymp x^{-r} l(x)$, *where* $l(x)$ *is a slowly varying function at infinity. Then, for any* $x > 0$,

i) $E\big(|X| I(|X| > x)\big) \asymp x^{1-r} l(x)$;

ii) $E\big(|X|^2 I(|X| \leq x)\big) \asymp x^{2-r} l(x)$.

Lemma 4 (Markov's inequality, [6]). *Let* $r > 0$ *and assume that* $E(|X|^r) < \infty$. *Then, for any* $x > 0$,

$$P(|X| > x) \leq \frac{E(|X|^r)}{x^r}.$$

Lemma 5. [6] *Let* $r > 0$. *Suppose that* $E(|X|^r) < \infty$ *and* $E(|Y|^r) < \infty$. *Then,*

$$E(|X + Y|)^r \leq C_r[E(|X|^r) + E(|Y|^r)],$$

where C_r *is a positive constant depending only on* r.

Lemma 6. [7] *Let* $1 < p < 2$ *and let* $\{X_n, n \geq 1\}$ *be a sequence of pairwise independent random variables with* $E(X_n) = 0$ *and* $E|X_n|^p < \infty$ *for all* $n \geq 1$. *Then,*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq \left(\frac{8}{2-p} + \frac{4}{p-1}\right) \sum_{i=1}^{n} E|X_i|^p,$$

and

$$E \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} X_i\right|^p$$
$$\leq \left(\left(\frac{\ln 2n}{\ln 2}\right)^2 \frac{8}{2-p} + \frac{4}{p-1}\right) \sum_{i=1}^{n} E|X_i|^p.$$

Lemma 7 (Proposition 1.3.6 in [8]) *If* $l(x)$ *is a slowly varying function and* $\alpha > 0$, *then*

$$x^{\alpha} l(x) \to +\infty \ \text{ and } \ x^{-\alpha} l(x) \to 0 \ \text{ as } x \to +\infty.$$

Lemma 8 (Proposition 1.5.10 in [8]) *If* $l(x)$ *is a slowly varying function and* $\alpha < -1$, *then* $\int_x^{\infty} t^{\alpha} l(t)dt$ *converges and*

$$\frac{x^{\alpha+1} l(x)}{\int_x^{\infty} t^{\alpha} l(t)dt} \to -\alpha - 1 \ \text{ as } x \to \infty.$$

Lemma 9 (Theorem 1.5.4 in [8]) *A (positive, measurable) function* $l(x)$ *is slowly varying if and only if, for every* $\alpha > 0$, *there exists a non-decreasing function* $\phi(x)$ *and non-increasing function* $\psi(x)$ *with*

$$x^{\alpha} l(x) \sim \phi(x),\ x^{-\alpha} l(x) \sim \psi(x) \text{ as } x \to \infty.$$

2. Main results

Theorem 1. *Let* $1 < r < 2$, $0 < p < r$, *and let* $\{X_n, n \geq 1\}$ *be a sequence of pairwise independent random variables with mean zero that are stochastically dominated by a random variable X satisfying* $P(|X| > x) \asymp x^{-r} l(x)$, *where* $l(x)$ *is a slowly varying function at infinity. Suppose that* $\{a_{ni}; 1 \leq i \leq n, n \geq 1\}$ *is a triangular array of real numbers such that*

$$\sum_{i=1}^{n} a_{ni}^2 = O(n).$$

Then,

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ni} X_i \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Proof: For every $n \geq 1$ and $1 \leq i \leq n$, we define

$$Y_{ni} = X_i I\big(|X_i| \leq n^{1/p}\big);\ Z_{ni} = X_i I\big(|X_i| > n^{1/p}\big);$$
$$S_n = \sum_{i=1}^{n} a_{ni}[Y_{ni} - E(Y_{ni})]\,;\, S_n' = \sum_{i=1}^{n} a_{ni}[Z_{ni} - E(Z_{ni})].$$

Clearly, for every $n \geq 1$ one has

$$X_i = Y_{ni} + Z_{ni}, \sum_{i=1}^{n} a_{ni} X_i = S_n + S_n'.$$

For any $\varepsilon > 0$ and $n \geq 1$, we have

$$\mathrm{P}\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon n^{1/p}\right)$$
$$\leq P\left(|S_n| > \frac{\varepsilon n^{1/p}}{2}\right) + P\left(|S_n'| > \frac{\varepsilon n^{1/p}}{2}\right) := I_1 + I_2$$

It suffices to show that

$$I_1 \to 0 \ \text{ and } \ I_2 \to 0 \ \text{ as } n \to \infty.$$

By Markov's inequality,

$$I_1 = P\left(|S_n| > \frac{\varepsilon n^{1/p}}{2}\right) \leq E(|S_n|^2) \Big/ \left(\frac{\varepsilon n^{1/p}}{2}\right)^2$$
$$= \frac{4}{\varepsilon^2 n^{2/p}} \sum_{i=1}^{n} a_{ni}^2 E[Y_{ni} - E(Y_{ni})]^2$$
$$\leq \frac{4}{\varepsilon^2 n^{2/p}} \sum_{i=1}^{n} a_{ni}^2 E(Y_{ni}^2)$$
$$= \frac{4}{\varepsilon^2 n^{2/p}} \sum_{i=1}^{n} a_{ni}^2 E\left(|X_i|^2 I\big(|X_i| \leq n^{1/p}\big)\right).$$

Applying Lemmas 2, 3, and 7 together with the assumptions of the theorem, we obtain

$$\begin{aligned}
I_1 &\le \frac{4}{\varepsilon^2 n^{2/p}}\sum_{i=1}^{n} a_{ni}^2\left(E\left(|X|^2 I\{|X|\le n^{1/p}\}\right)+n^{2/p}P\left(|X|>n^{1/p}\right)\right)\\
&\le \frac{Cn}{\varepsilon^2 n^{2/p}}\left(E\left(|X|^2 I\{|X|\le n^{1/p}\}\right)+n^{2/p}P\left(|X|>n^{1/p}\right)\right)\\
&\le \frac{Cn}{\varepsilon^2 n^{2/p}}\left[\left(n^{1/p}\right)^{2-r} l\left(n^{1/p}\right)+n^{2/p}P\left(|X|>n^{1/p}\right)\right]\\
&= \frac{Cl\left(n^{1/p}\right)}{\varepsilon^2 n^{r/p-1}}+\frac{Cn}{\varepsilon^2}P\left(|X|>n^{1/p}\right)\\
&\le \frac{Cl\left(n^{1/p}\right)}{\varepsilon^2 n^{r/p-1}}+\frac{Cn}{\varepsilon^2}\left(n^{1/p}\right)^{-r} l\left(n^{1/p}\right)\\
&\le Cn^{1-r/p} l(n^{1/p}) \to 0, \text{ khi } n\to\infty.
\end{aligned}$$

Obviously, for every $n \ge 1$

$$\sum_{i=1}^{n}|a_{ni}| \le \left(n\sum_{i=1}^{n}|a_{ni}|^2\right)^{1/2} \le Cn.$$

Note that by Lemmas 7 and 8, for any $q \in (0, r)$

$$\begin{aligned}
E|X|^q &= -\int_0^\infty |x|^q dP(|X|>x)\\
&= q\int_0^\infty |x|^{q-1}P(|X|>x)dx\\
&\le C\int_0^\infty |x|^{q-r-1} l(x)dx < \infty.
\end{aligned}$$

Therefore, similarly, by Markov's inequality and Lemmas 2, 3, and 7, we also have

$$\begin{aligned}
I_2 &= P\left(|S'_n| > \frac{\varepsilon n^{1/p}}{2}\right) \le E(|S'_n|)\Big/\left(\frac{\varepsilon n^{1/p}}{2}\right)\\
&\le \frac{2}{\varepsilon n^{1/p}}E\left(\sum_{i=1}^{n}|a_{ni}(Z_{ni}-E(Z_{ni}))|\right)\\
&\le \frac{C}{\varepsilon n^{1/p}}\sum_{i=1}^{n}E|a_{ni}Z_{ni}|\\
&= \frac{C}{\varepsilon n^{1/p}}\sum_{i=1}^{n}E\left|a_{ni}X_i I\left(|X_i|>n^{1/p}\right)\right|\\
&\le \frac{C}{\varepsilon n^{1/p}}\sum_{i=1}^{n}|a_{ni}|E\left(|X| I\{|X|>n^{1/p}\}\right)\\
&\le \frac{Cn}{\varepsilon n^{1/p}}\left(n^{1/p}\right)^{1-r} l\left(n^{1/p}\right)\\
&= \frac{C\, l\left(n^{1/p}\right)}{\varepsilon n^{r/p-1}} \to 0 \text{ khi } n\to\infty.
\end{aligned}$$

The theorem is proved. ■

Theorem 2. *Let* $1 < r < 2$, $0 < p < r$, *and let* $\{X_n; n \ge 1\}$ *be a sequence of pairwise independent random variables with mean zero that are stochastically dominated by a random variable* X *satisfying* $P(|X| > x) \asymp x^{-r}l(x)$, *where* $l(x)$ *is a slowly varying function at infinity. Suppose that* $\{a_{ni}; 1 \le i \le n, n \ge 1\}$ *is a triangular array of real numbers such that*

$$\sum_{i=1}^{n} a_{ni}^r = O(n).$$

Then,

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n} a_{ni}X_i \to 0\ (a.s.) \text{ as } n\to\infty.$$

Proof: For $n \ge 1$ and $1 \le i \le n$, define Y_{ni}, Z_{ni} as in the proof of Theorem 1. For $1 \le k \le n$, set $S_{nk} = \sum_{i=1}^{k} a_{ni}[Y_{ni} - E(Y_{ni})]$ and $S'_{nk} = \sum_{i=1}^{k} a_{ni}[Z_{ni} - E(Z_{ni})]$. By Lemma 1, it suffices to show that for every $\varepsilon > 0$,

$$J := \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/p}\right) < \infty.$$

To prove this, it is enough to show that for every $\varepsilon > 0$

$$J_1 := \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{P}\left(\max_{1\le k\le n}|S_{nk}| > \varepsilon n^{1/p}/2\right) < \infty$$

and

$$J_2 := \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{P}\left(\max_{1\le k\le n}|S'_{nk}| > \varepsilon n^{1/p}/2\right) < \infty.$$

Applying Hölder's inequality, for every $n \ge 1$

$$\sum_{i=1}^{n}|a_{ni}| \le \left(\sum_{i=1}^{n}1\right)^{1-1/r}\left(\sum_{i=1}^{n}|a_{ni}|^r\right)^{1/r} \le Cn.$$

Note that by Lemma 9, for any $\alpha > 1$ and sufficiently large n, the function $f(x) = x^{-\alpha}l(x)$ can be considered as a non-increasing function in $[n; +\infty)$. Thus, Lemma 8 implies that

$$\begin{aligned}
\sum_{k=n}^{\infty}\frac{l(k)}{k^\alpha} &= \sum_{k=n}^{\infty}k^{-\alpha}l(k)\\
&\le \sum_{k=n}^{\infty}\int_k^{k+1}x^{-\alpha}l(x)\,dx = \int_n^\infty x^{-\alpha}l(x)dx\\
&\le C.n^{-\alpha+1}l(n) = \frac{Cl(n)}{n^{\alpha-1}}.
\end{aligned}$$

Consider J_1. Clearly, for n large enough $\ln^2(2n) \le C\ln^2(n)$. By Markov's inequality and Lemmas 2 and 6, we obtain

$$\begin{aligned}
J_1 &\le \sum_{n=1}^{\infty}\frac{2^{\mathrm{r}}}{\varepsilon^r n^{1+r/p}}\mathrm{E}\left(\max_{1\le k\le n}|S_{nk}|^r\right)\\
&\le \sum_{n=1}^{\infty}\frac{\mathrm{C}(\ln 2n)^2}{n^{1+r/p}}\sum_{\mathrm{i}=1}^{\mathrm{n}}|a_{ni}|^r\,\mathrm{E}(|Y_{ni}-EY_{ni}|^r)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln 2n)^2}{n^{1+r/p}}\sum_{i=1}^{n}|a_{ni}|^r\,E(|Y_{ni}|^r)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln 2n)^2}{n^{1+r/p}}\sum_{i=1}^{n}|a_{ni}|^r\,E\left(|X_i|^r I\{|X_i|\le n^{1/p}\}\right)\\
&\le C\sum_{n=1}^{\infty}(\ln 2n)^2\,P(|X|>n^{1/p})
\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty}\frac{(\ln 2n)^2}{n^{r/p}}E\big(|X|^r I\{|X|\le n^{1/p}\}\big)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln 2n)^2}{n^{r/p}}l(n^{1/p})\\
&\quad+C\sum_{n=1}^{\infty}\frac{(\ln 2n)^2}{n^{r/p}}\sum_{k=1}^{n}E\big(|X|^r I\{(k-1)^{1/p}<|X|\\
&\qquad\qquad\le k^{1/p}\}\big)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{r/p}}l(n^{1/p})\\
&\quad+C\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{r/p}}\sum_{k=1}^{n}E\big(|X|^r I\{(k-1)^{1/p}<|X|\\
&\qquad\qquad\le k^{1/p}\}\big)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{r/p}}l(n^{1/p})\\
&\quad+C\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{(\ln n)^2}{n^{r/p}}E\big(|X|^r I\{(k-1)^{1/p}<|X|\\
&\qquad\qquad\le k^{1/p}\}\big)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{r/p}}l(n^{1/p})\\
&\quad+C\sum_{k=1}^{\infty}\frac{(\ln k)^2}{k^{r/p-1}}E\big(|X|^r I\{(k-1)^{1/p}<|X|\le k^{1/p}\}\big)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{r/p}}l(n^{1/p})\\
&\quad+C\sum_{k=1}^{\infty}\frac{(\ln k)^2}{k^{r/p-1}}E\big(|X|^p|X|^{r-p}I\{(k-1)^{1/p}\\
&\qquad\qquad<|X|\le k^{1/p}\}\big)\\
&\le C\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{r/p}}l(n^{1/p})\\
&\quad+C\sum_{k=1}^{\infty}(\ln k)^2\,E\big(|X|^p I\{(k-1)^{1/p}<|X|\le k^{1/p}\}\big)\\
&\le \sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{r/p}}l(n^{1/p})+CE(|X|^p\ln^2(1+|X|)).
\end{aligned}
$$

By Lemma 7, for sufficiently small $\delta>0$ such that $(r-\delta)/p>1$ and for sufficiently large n_0, we have

$$
\begin{aligned}
&\sum_{n=n_0}^{\infty}\frac{(\ln n)^2}{n^{r/p}}l\big(n^{1/p}\big)\le\sum_{n=n_0}^{\infty}\frac{(\ln n)^2}{n^{r/p}}\big(n^{1/p}\big)^{\delta}\\
&=\sum_{n=n_0}^{\infty}\frac{(\ln n)^2}{n^{(r-\delta)/p}}<\infty.
\end{aligned}
$$

On the other hand, for sufficiently large x,

$$x^p\ln^2(1+x)\,P(|X|>x)\le Cx^{p-r}\ln^2(1+x)\,l(x)$$

and hence, by Lemma 7,

$$\lim_{x\to\infty}x^p\ln^2(1+x)\,P(|X|>x)=0.$$

Since $\ln^2(x)l(x)$ is also slowly varying, by applying Lemmas 8, it follows that for any sufficiently large constant $A>0$,

$$
\begin{aligned}
&E(|X|^p\ln^2(1+|X|)\,I(|X|>A))\\
&\quad=-\int_A^{\infty}x^p\ln^2(1+x)\,dP(|X|>x)\\
&\quad=A^p\ln^2(1+A)\,P(|X|>A)+\\
&\quad\int_A^{\infty}x^{p-1}\ln(1+x)\left(p\ln(1+x)+\frac{2x}{1+x}\right)P(|X|>x)dx\\
&\quad\le A^p\ln^2(1+A)\,P(|X|>A)\\
&\qquad+C\int_A^{\infty}x^{p-1}\ln^2(1+x)\,P(|X|>x)dx\\
&\quad\le A^p\ln^2(1+A)\,P(|X|>A)\\
&\qquad+C\int_A^{\infty}x^{p-1-r}\ln^2(x)\,l(x)dx<\infty.
\end{aligned}
$$

Therefore, $J_1<\infty$. Next, we shall prove that $J_2<\infty$. By Markov's inequality together with Lemmas 2 and 3,

$$
\begin{aligned}
J_2&\le\sum_{n=1}^{\infty}\frac{2}{\varepsilon n^{1+1/p}}\mathrm{E}\left(\max_{1\le k\le n}|S'_{nk}|\right)\\
&\le\sum_{n=1}^{\infty}\frac{2}{\varepsilon n^{1+1/p}}\sum_{i=1}^{n}E(|a_{ni}(Z_{ni}-E(Z_{ni}))|)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{1+1/p}}\sum_{i=1}^{n}E(|a_{ni}Z_{ni}|)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{1+1/p}}\sum_{i=1}^{n}|a_{ni}|E(|X_i|I\big(|X_i|>n^{1/p}\big))\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{1+1/p}}\sum_{i=1}^{n}|a_{ni}|E(|X|I(|X|>n^{1/p}))\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\big(n^{1/p}\big)^{1-r}l\big(n^{1/p}\big)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n^{r/p}}l\big(n^{1/p}\big)<\infty.
\end{aligned}
$$

This completes the proof. ■

Corollary: *Let* $1<r<2$, $0<p<r$ *and let* $\{X_n,n\ge 1\}$ *be a sequence of pairwise independent random variables with mean zero that are stochastically dominated by a random variable X satisfying* $P(|X|>x)\asymp x^{-r}l(x)$, *where* $l(x)$ *is a slowly varying function at infinity. Then,*

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}X_i\ \to 0\ (a.s.)\ \text{ as }\ n\to\infty.$$

3. Application

Consider the regression model:

$$y_{ni}=f(x_{ni})+\varepsilon_{ni},1\le i\le n,\tag{1}$$

where x_{ni} are known fixed designed points in the compact set $A\subset\boldsymbol{R}^m$, $f(x)$ is an unknown regression function defined on the set A, ε_{ni} are random errors. The nonparametric regression estimator for the model is defined by

$$\hat{f}_n(x)=\sum_{i=1}^{n}W_{ni}(x)\,y_{ni},$$

where $W_{ni}(x) = W_i(x, x_{n1}, x_{n2}, \dots, x_{nn})$ are the weight functions. For any $x \in A$, assume that the weight functions $W_{ni}(x)$ satisfy the following conditions as $n \to \infty$:

$$(A1) \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right| = o(1),$$

$$(A2) \left| \sum_{i=1}^{n} W_{ni}(x) \right| = O(1),$$

$(A3)$

$$\sum_{i=1}^{n} |W_{ni}(x)| |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > a) = o(1),$$

for any $a > 0$.

By similar arguments to those used in the proof of Theorem 4.2 in [3], together with an application of Theorems 1 and 2, we obtain the following result.

Theorem 3. *Let* $1 < r < 2,\ 0 < p < r$. *In the model (1), assume that for* $n \geq 1$, $\{\varepsilon_{ni}, 1 \leq i \leq n\}$ *is a sequence of pairwise independent random variables with mean zero that are stochastically dominated by a random variable* ε *satisfying* $P(|\varepsilon| > x) \asymp x^{-r} l(x)$, *where* $l(x)$ *is a slowly varying function at infinity. Denote the set of all continuity points of the function* $f(x)$ *by* $C(f)$.

i) If

$$\sum_{i=1}^{n} W_{ni}^2(x) = O\left(n^{1-2/p}\right), \text{ as } n \to \infty,$$

then for any $x \in C(f)$

$$\hat{f}_n(x) \xrightarrow{P} f(x) \text{ as } n \to \infty.$$

ii) If

$$\sum_{i=1}^{n} W_{ni}^r(x) = O\left(n^{1-r/p}\right), \text{ as } n \to \infty,$$

then for any $x \in C(f)$

$$\hat{f}_n(x) \longrightarrow f(x)\ (a.s.) \text{ as } n \to \infty.$$

4. Conclusion

This paper establishes limit theorems, including the weak and strong laws of large numbers, for sequences of weighted sums of pairwise independent random variables that are stochastically dominated by a heavy-tailed random variable. These results extend the classical Marcinkiewicz–Zygmund theorems by replacing the identical distribution assumption with a stochastic domination condition. Moreover, the results generalize those of Xuan et al. [3] to the class of pairwise independent random variables with heavy-tailed distributions. They can be applied to study the convergence of weighted sums of random variables, such as a nonparametric regression estimator widely used in statistics and data science.

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