# NO NONTRIVIAL BAUMSLAG-SOLITAR RELATIONS IN EXTENDED ADMISSIBLE GROUPS

**Phan Quang Nhu Anh[1], Phan Van Thien[2], Chu Van Tiep[1]**

[1]*The University of Danang - University of Science and Education, Vietnam*

[2]*University of Education – The University of Hue, Vietnam*

*Corresponding author: pqnanh@ued.udn.vn

(Received: May 11, 2026; Revised: June 12, 2026; Accepted: June 15, 2026)

DOI: 10.31130/ud-jst.2026.24(6A).334E

**Abstract -** In this work, we analyze the structural attributes of extended admissible groups, which constitute a diverse family of finitely generated groups. These groups serve as a generalization for both the graph-of-groups decompositions linked to non-geometric 3-manifolds and the standard admissible groups introduced by Croke and Kleiner. Our principal finding demonstrates that extended admissible groups do not support any nontrivial Baumslag–Solitar relations. This result generalizes a classical theorem establishing that the fundamental groups of 3-manifolds are free from such relations. By blending geometric group theory methodologies with Bass–Serre theory, our proof examines the group action on the corresponding Bass–Serre tree alongside the behavior of relatively hyperbolic vertex groups. This conclusion enhances our understanding of rigidity phenomena within relatively hyperbolic settings and offers additional validation for the quasi-isometric rigidity of extended admissible groups.

**Key words -** Baumslag-Solitar relations; extended admissible groups; relatively hyperbolic groups; graphs of groups; quasi-isometric rigidity.

## 1. Introduction

The classical presentation

$$BS(m,n) = \langle a,b \mid b\, a^{\mathrm{m}}\, b^{-1} = a^{\mathrm{n}} \rangle$$

specifies the Baumslag–Solitar group, where the integers $m$ and $n$ are assumed to be non-zero throughout this discussion.

**Definition 1.1.** Supposing $K$ is a torsion-free group, a pair of elements $r,s \in K \setminus \{1\}$ is said to satisfy a *nontrivial Baumslag–Solitar relation* if there are non-zero integers $m$ and $n$ with $m \neq \pm n$ such that the identity

$$r\, s^{\mathrm{m}}\, r^{-1} = s^{\mathrm{n}}$$

is satisfied.

A well-known principle in low-dimensional topology asserts that nontrivial Baumslag–Solitar relations cannot exist within the fundamental groups of 3-manifolds. This paper aims to extend this structural property to a broader algebraic setting involving extended admissible groups, a class originally presented in [1]. This collection of groups broadens the scope of graph of groups decompositions originating from non-geometric 3-manifolds, expanding upon the admissible groups proposed by Croke and Kleiner [2].

We offer here a preliminary summary regarding how extended admissible groups are structured. A fully rigorous formulation can be found in Definition 2.2.

Consider a non-geometric 3-manifold $M$. Utilizing the torus decomposition, one obtains a unique (up to isotopy) minimal, nonempty family $\mathcal{T} \subset M$ consisting of disjoint essential tori. This collection partitions $M$ into distinct components, where each sub-manifold $M_{\mathrm{v}}$ of $M \setminus \mathcal{T}$ is referred to as a piece. Based on geometric decomposition theory, every piece is either Seifert fibered or hyperbolic. This topological configuration naturally induces a graph of groups structure $\mathcal{G}$ for $\pi_1(M)$ over an underlying graph $\Gamma$ as follows:

- Each separate piece $M_{\mathrm{v}}$ is associated with a specific vertex $v \in V\Gamma$, having $\pi_1(M_{\mathrm{v}})$ as its vertex group.

- Whenever a torus $T_{\mathrm{e}} \in \mathcal{T}$ forms the shared boundary between two pieces $M_{\mathrm{u}}$ and $M_{\mathrm{v}}$, there exists a corresponding edge $e \in E\Gamma$ that links vertices $u$ and $v$.

- The edge group assigned to e is isomorphic to $\mathbb{Z}^2$ (given by $\pi_1(T_{\mathrm{e}})$), with the inclusion maps directly inducing the edge monomorphisms.

Croke and Kleiner [2] formulated the framework of admissible groups to provide a graph of groups setting that captures the structural properties of graph manifolds. In the broader context of an extended admissible group, this architectural design is generalized further: the Seifert fibered components are substituted by arbitrary $\mathbb{Z}$-by-hyperbolic groups, whereas the hyperbolic 3-manifold pieces are replaced by toral relatively hyperbolic groups.

The main contribution of this research is formulated in the following theorem.

**Theorem 1.2.** If $K$ is an extended admissible group, then $K$ does not admit any nontrivial Baumslag-Solitar relations.

## 2. Preliminary

This section reviews key definitions and essential geometric group theory concepts required for our subsequent derivations.

Relatively hyperbolic groups can be characterized through several equivalent frameworks. In this paper, we utilize the formulation developed by Bowditch [3].

Let $K$ be a finitely generated group equipped with a finite collection of subgroups $\mathcal{P}$. Fix a finite generating set $S$ for $K$, and let $\Gamma(K,S)$ be the associated Cayley graph. We consider this graph under the standard word metric $d_S$, denoting the word length of an element g as $|g|_S = d_{S(1,g)}$.

Let $\mathcal{P} = \{gP: g \in K, P \in \mathcal{P}\}$ represent the set of all peripheral cosets. Starting from $\Gamma(K,S)$, the coned-off Cayley graph, denoted by $\hat{K}(\mathcal{P})$, is constructed by the following mechanism: for each peripheral coset $P \in \mathcal{P}$, we introduce a distinct cone vertex $c(P)$ and connect it to every element in $P$ using half-edges. A peripheral edge is formed by joining two such half-edges at their shared cone point. The metric on this newly formed coned-off graph is denoted by $\hat{d}_S$.

The pair $(K,\mathcal{P})$ is called relatively hyperbolic provided that the coned-off Cayley graph $\hat{K}(\mathcal{P})$is hyperbolic and fulfills the fineness criterion, which dictates that any given edge can belong to only a finite number of simple cycles of a uniformly bounded length.

**Definition 2.1.** A graph of groups $\mathcal{G} = (\Gamma, \{K_v\}, \{K_e\}, \{\tau_e\})$ comprises the following components:

1. A base graph $\Gamma$.

2. A vertex group $K_v$ corresponding to each vertex $v \in V\Gamma$.

3. An edge group $K_e \leq K_{e^-}$ corresponding to each edge $e \in E\Gamma$.

4. An edge monomorphism consisting of an isomorphism $\tau_e$: $K_e \to K_{e^+}$ for each $e \in E\Gamma$, satisfying the condition $\tau_e{}^{-1} = \tau_{e^-}$.

Following the classic approach in [4], the fundamental group $\pi_1(\mathcal{G})$ of the graph of groups $\mathcal{G}$ is defined in the standard way.

We now restate the formal characterization of extended admissible groups as introduced in [1].

**Definition 2.2.** A group $K$ is defined as an *extended admissible group* if it can be represented as the fundamental group of a graph of groups $\mathcal{G}$ that complies with the following conditions:

1. The underlying graph $\Gamma$ of $\mathcal{G}$ is finite, connected, contains at least one edge, and all of its edge groups are virtually $\mathbb{Z}^2$.

2. Each vertex group $K_v$ falls into one of the two categories detailed below:

**(a) Type $\mathcal{B}$:** $K_v$ contains an infinite cyclic normal subgroup $Z_v \lhd K_v$ such that the quotient group $Q_v = K_v / Z_v$ is non-elementary hyperbolic. We call $Z_v$ the kernel and $Q_v$ the hyperbolic quotient of $K_v$.

**(b) Type $\mathcal{H}$:** $K_v$ is relatively hyperbolic with respect to a collection $\mathbb{P}_v$ of virtually $\mathbb{Z}^2$-subgroups. In addition, all edge groups incident to $K_v$ are embedded in $\mathbb{P}_v$, and $K_v$ does not possess a splitting relative to $\mathbb{P}_v$ over any subgroup belonging to an element of $\mathbb{P}_v$.

3. For any vertex group $K_v$, given any edges $e$, $e' \in \text{link}(v)$ and an element $g \in K_v$, the conjugate subgroup $g\, K_e\, g^{-1}$ is commensurable with $K_{e'}$ if and only if $e = e'$ and $g \in K_e$.

4. For every edge group $K_e$ whose adjacent vertex groups $K_{e'}$ and $K_{e^+}$ are both of Type $\mathcal{B}$, the subgroup generated by $\tau_e(Z_{e^-} \cap K_e)$ and $Z_{e^+} \cap K_e$ has a finite index in $K_e$.

### 3. No nontrivial Baumslag–Solitar relations

**Lemma 3.1.** Any vertex group belonging to Type $\mathcal{B}$ is free of nontrivial Baumslag–Solitar relations.

*Proof.* Let $K$ be a Type $\mathcal{B}$ vertex group. By definition, $K$ admits a normal subgroup $Z(K) \cong \mathbb{Z}$ such that the quotient

$$Q := K / Z(K)$$

is non-elementary hyperbolic. Let $\pi: K \to Q$ denote the canonical projection map.

Supposing a nontrivial Baumslag–Solitar relation holds in $K$, its image under $\pi$ would yield a corresponding nontrivial Baumslag–Solitar relation in $Q$. However, non-elementary hyperbolic groups cannot contain such relations (see [5]). Consequently, the relation can only persist if $r \in Z(K)$ or $s \in Z(K)$.

Assuming $r \in Z(K)$ without loss of generality, the relation $r\, s^m\, r^{-1} = s^n$ simplifies to $s^m = s^n$ because $r$ lies in the center and commutes with all elements. This implies $m = n$, directly contradicting the hypothesis that $m \neq \pm n$. □

**Theorem 3.2.** Let $K$ be an extended admissible group. Then no nontrivial Baumslag–Solitar relation holds in $K$.

*Proof.* To obtain a contradiction, suppose there exist elements $r, s \neq 1$ in $K$ satisfying a nontrivial relation $r\, s^m\, r^{-1} = s^n$ for non-zero integers $m, n$ with $m \neq \pm n$. Let $T$ be the Bass–Serre tree associated with the graph of groups structure of $K$ (cf. [7]).

If $s$ does not fix any vertex in $T$, it acts as a translation on a unique invariant axis $A$. For any point $a \in A$, the translation length is strictly positive, defined by $\ell(s) = d(a, s \cdot a) > 0$. Because the translation length function is translation-additive for powers and invariant under conjugation, we obtain:

$$\ell(s^m) = \ell(r\, s^m\, r^{-1}) = \ell(s^n)$$

This leads to the equation $|m|\ell(s) = |n|\ell(s)$. Since $\ell(s) > 0$, it follows that $m = \pm n$, which contradicts our assumption.

Therefore, $s$ must fix some vertex $v \in V(T)$. For any element $g \in H$, it follows that:

$$d(g^{-1} \cdot v, s\, g^{m(g)} g^{-1} \cdot v) = d(v, g\, s^{m(g)}\, g^{-1} \cdot v)$$
$$= d(v, s\, g^{n(g)} \cdot v) = 0$$

Since the nontrivial element $s\, g^{m(g)} \neq 1$ fixes both $v$ and $g^{-1} \cdot v$, applying [1, Lemma 2.29] guarantees that $d(v, g^{-1} \cdot v) \leq 2$. This shows that the orbit $H \cdot v$ stays within a bounded neighborhood of $v$, meaning it is finite. Consequently, the group $H$ must stabilize some vertex $w \in V(T)$.

By Lemma 3.1, vertex groups of Type $\mathcal{B}$ cannot host a nontrivial Baumslag–Solitar relation. Thus, the stabilizer must be of Type $\mathcal{H}$. However, [6, Corollary 4.2] states that the Baumslag–Solitar group $BS(m, n)$ must embed as a parabolic subgroup of this stabilizer, forcing it to be virtually $\mathbb{Z}^2$. This directly contradicts the condition $m \neq \pm n$. □

## 4. Conclusion

In this paper, we proved that extended admissible groups do not admit any nontrivial Baumslag–Solitar relation. This extends the corresponding result for fundamental groups of 3–manifolds to a wider class of groups built from relatively hyperbolic and $\mathbb{Z}$–by–hyperbolic vertex groups. The proof combines Bass–Serre theory with properties of relatively hyperbolic groups and vertex groups of type $\mathcal{B}$ and type $\mathcal{H}$. Our result provides further evidence of the rigidity phenomena and quasi-isometric rigidity of extended admissible groups.

**Acknowledgments:** The author was partially supported by the National Key Program for the development of Mathematics in the period from 2021 to 2030 under grant number B2024-CTT-04.

## REFERENCES

[1] A. Margolis and H. Nguyen, "Quasi-isometric rigidity of extended admissible groups", *arXiv.org*, 2024. [Online]. Available: https://arxiv.org/abs/2401.03635 [Accessed May 14, 2026].

[2] C. Croke and B. Kleiner, "The geodesic flow of a nonpositively curved graph manifold", *Geometric and Functional Analysis*, vol. 12, no. 3, pp. 479–545, 2002.

[3] B. Bowditch, "Relatively hyperbolic groups", *International Journal of Algebra and Computation*, vol. 22, no. 3, Art. no. 1250016, 2012.

[4] P. Scott and T. Wall, "Topological methods in group theory", in *Homological Group Theory* (Proc. Sympos., Durham, 1977). Cambridge, UK: Cambridge University Press, 1979, pp. 137–203.

[5] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*. Berlin, Germany: Springer–Verlag, 1999.

[6] D. Osin, "Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems", *Memoirs of the American Mathematical Society*, vol. 179, no. 843, 2006.

[7] J. P. Serre, *Trees*, Translated from the French by John Stillwell. Berlin: Springer-Verlag, 1980.