

THE REGULARITY INDEX OF $n + 4$ DOUBLE POINTS IN PROJECTIVE SPACES $\mathbb{P}^n$ FOR $n = 2$ AND $n = 3$

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Abstract - Investigating the regularity index of fat point schemes in higher-dimensional projective spaces remains a challenging problem and often relies on explicit computations in lower-dimensional cases. Such results provide valuable evidence for understanding the general behavior of the regularity index and for formulating broader conjectures. In this paper, we completely determine the regularity index of a set of $n + 4$ double points not lying on two lines in the projective space $\mathbb{P}^n$ for the two fundamental dimensions $n = 2$ and $n = 3$. More precisely, we establish explicit formulas for the regularity index in both cases. These results contribute to the study of fat point schemes and provide a foundation for future investigations in higher-dimensional projective spaces.

Key words - Regularity index; fat point schemes; double points; projective spaces; Hilbert function

1. Introduction

Let K be an algebraically closed field and let $\mathbb{P}^n$ denote the n -dimensional projective space over K . Put $R = K[X_0, \dots, X_n]$. Let $P_1, \dots, P_s$ be pairwise distinct points in $\mathbb{P}^n$, and let $m_1, \dots, m_s$ be positive integers. For each $i = 1, \dots, s$, denote by $\wp_i$ the homogeneous prime ideal corresponding to the point P_i . Consider the ideal

$$I_Z = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}.$$

This ideal consists of all homogeneous polynomials whose partial derivatives of order less than m_i vanish at P_i for every $i = 1, \dots, s$. Hence, I_Z defines the zero-dimensional scheme

$$Z = m_1 P_1 + \dots + m_s P_s,$$

which is usually called a fat point scheme in $\mathbb{P}^n$.

The quotient ring R/I_Z is the homogeneous coordinate ring associated with Z . It admits the graded decomposition

$$R/I_Z = \bigoplus_{t \geq 0} (R/I_Z)_t,$$

and its multiplicity is given by

$$e(R/I_Z) = \sum_{i=1}^s \binom{m_i + n - 1}{n}.$$

For every nonnegative integer t , the graded component $(R/I_Z)_t$ is a finite-dimensional K -vector space. The Hilbert function of R/I_Z is defined by

$$H_{R/I_Z}(t) = \dim_K(R/I_Z)_t.$$

This function increases until it attains the value $e(R/I_Z)$, after which it remains constant.

The integer

$$\text{reg}(R/I_Z) = \min\{t \in \mathbb{N} \mid H_{R/I_Z}(t) = e(R/I_Z)\}.$$

is called the regularity index of R/I_Z . We also refer to

this quantity as the regularity index of the fat point scheme Z and write $\text{reg}(Z)$. Not only determining effective upper bounds but also computing the exact values of $\text{reg}(Z)$ has been the subject of considerable research; these results can be found, for example, in [1]–[8] and the references therein.

Investigating the regularity index of fat point schemes in higher-dimensional projective spaces is often reduced to the analysis of corresponding configurations in lower-dimensional projective subspaces. In particular, the cases $n = 2$ and $n = 3$ play a fundamental role, since arguments in higher dimensions frequently rely on a detailed understanding of these lower-dimensional configurations. Motivated by this fact, in this paper we completely determine the regularity index of a set of $n + 4$ double points not lying on two lines in the projective space $\mathbb{P}^n$ for the two cases $n = 2$ and $n = 3$. These results provide a complete treatment of the problem in the lowest nontrivial dimensions and establish a foundation for investigating the corresponding problem in higher-dimensional projective spaces.

The main result of this paper is presented in Theorem 3.2.

2. Preliminary

In the process of proving the main results, we utilize the lemmas already proven in [2], [4], [5], [7].

The following lemma is used to calculate the value of $\text{reg}(Z)$ when the set of points lies on a line.

Lemma 2.1. ([2], Corollary 2.3). *If $Z = m_1 P_1 + \dots + m_s P_s$ is an arbitrary set of fat points in $\mathbb{P}^n$, then $\text{reg}(Z) = m_1 + \dots + m_s - 1$ if and only if the points $P_1, \dots, P_s$ lie on a line.*

Let $\{1, \dots, s\}$ be the set of indices and $\{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset of indices. We call $Y = m_{i_1} P_{i_1} + \dots + m_{i_r} P_{i_r}$ a fat point subset of $Z = m_1 P_1 + \dots + m_s P_s$. The following lemma helps compare the regularity index of a fat point subset with that of a given fat point set.

Lemma 2.2. ([4], Lemma 3.3). *Let $X = \{P_1, \dots, P_s\}$ be distinct points in $\mathbb{P}^n$ and $m_1, \dots, m_s$ be positive integers. Let $I = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$. If $Y = \{P_{i_1}, \dots, P_{i_r}\}$ is a subset of X and $J = \wp_{i_1}^{m_{i_1}} \cap \dots \cap \wp_{i_r}^{m_{i_r}}$, then $\text{reg}(R/J) \leq \text{reg}(R/I)$.*

Consequently, if $Z = m_1 P_1 + \dots + m_s P_s$ is the fat point set defined by the ideal I and $Y = m_{i_1} P_{i_1} + \dots +$

$m_{i_r} P_{i_r}$ is the fat point set defined by the ideal J , then we have $\text{reg}(Y) \leq \text{reg}(Z)$.

Lemma 2.3. ([5], Lemma 2.1). *Let $P_1, \dots, P_r, P$ be distinct points in $\mathbb{P}^n$ and $\wp$ be the ideal defined by the point P . If $m_1, \dots, m_r$ and a are positive integers, $J := \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$, and $I = J \cap \wp^a$, then*

$$\text{reg}(R/I) = \max\{a - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp^a))\}.$$

Lemma 2.4. ([5], Lemma 2.2). *Let $P_1, \dots, P_r$ be distinct points in $\mathbb{P}^n$ and $a, m_1, \dots, m_r$ be positive integers. Let $J = \wp_1^{m_1} \cap \dots \cap \wp_r^{m_r}$ and $\wp = (X_1, \dots, X_n)$. Then $\text{reg}\left(\frac{R}{J + \wp^a}\right) \leq b$ if and only if $X_0^{b-i} M \in J + \wp^{i+1}$ for every monomial M of degree i in $X_1, \dots, X_n$, $i = 0, \dots, a - 1$.*

Lemma 2.5. ([7], Theorem 2.5). *Let $X = \{P_1, \dots, P_{n+3}\}$ be a set of $n+3$ distinct non-degenerate points in $\mathbb{P}^n$ such that no $s+3$ points of X lie on an s -plane, $s < n$. Let $2 \leq m_1 \geq m_2 \geq \dots \geq m_{n+3}$ be positive integers and $Z = m_1 P_1 + \dots + m_{n+3} P_{n+3}$.*

Let

$$T_j = \max \left\{ \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j - \text{space} \right\}$$

and $T = \max\{T_j \mid j = 1, \dots, n\}$, Then $\text{reg}(Z) = T$, except in the case where: $n = 3, m_1 = m_2 = m_3 = m_4 = m_5 = m_6 + 1 = m$, no 3 points lie on a line, and there is a 2-plane passing through P_6 and 3 points of $\{P_1, \dots, P_5\}$. In this case, $T_1 = T_2 = 2m - 1$, $T_3 = 2m$, and $\text{reg}(Z) = 2m - 1$ or $\text{reg}(Z) = 2m$.

Lemma 2.6. ([3], Theorem 5.3). *If Z is any fat point subscheme of $\mathbb{P}^n$, then*

$$\text{reg}(Z) \leq \max\{T_j \mid j = 1, \dots, n\}$$

where

$$T_j = \max \left\{ \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j - \text{space} \right\}.$$

3. The regularity index of a set of $n + 4$ double points not lying on two lines in the projective space $\mathbb{P}^n$ for $n = 2$ and $n = 3$

We begin with the following lemma:

Lemma 3.1. *Let $X = \{P_1, \dots, P_s\}$ be a set of s distinct non-degenerate points ($s \leq n + 4$) in $\mathbb{P}^n$. Let $m_1, \dots, m_s$ be positive integers and consider the fat point set $Z = m_1 P_1 + \dots + m_s P_s$. Let*

$$T_j = \max \left\{ \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j - \text{space} \right\}$$

and

$$T = \max\{T_j \mid j = 1, \dots, n\}.$$

Then, if $T = T_1$, we have $\text{reg}(Z) = T$.

Proof.

Let d be the line passing through the points $P_{i_1}, P_{i_2}, \dots, P_{i_r}$ such that $T_1 = m_{i_1} + m_{i_2} + \dots + m_{i_r} - 1$.

Consider the fat point set $Y = m_{i_1} P_{i_1} + \dots + m_{i_r} P_{i_r}$. By Lemma 2.1, we have $\text{reg}(Y) = T_1 = T$. By Lemma 2.2, we have $\text{reg}(Z) \geq \text{reg}(Y) = T$.

Furthermore, Nagel and Trok proved in [3] that $\text{reg}(Z) \leq T$. Therefore, $\text{reg}(Z) = T$.

This completes the proof of Lemma 3.1.

Theorem 3.2. *Let $Z = 2P_1 + \dots + 2P_{n+4}$ be a fat point scheme of $n + 4$ double points not in linearly general position, not lying on two lines, and non-degenerate in the projective space $\mathbb{P}^n$, for $n = 2, 3$.*

Let

$$T_j = \max \left\{ \left\lfloor \frac{\sum_{l=1}^q m_{i_l} + j - 2}{j} \right\rfloor \mid P_{i_1}, \dots, P_{i_q} \text{ lie on a linear } j - \text{space} \right\}$$

and

$$T = \max\{T_j \mid j = 1, \dots, n\}.$$

Let

$$k_j = \min\{t \mid \exists t + 1 + j \text{ points lying on a } t - \text{space}\}, j = 1, 2, 3.$$

Then:

1) We have $\text{reg}(Z) = T - 1$ in the following cases:

(a) $n = 2$ and $k_1 = 1$;

(b) $n = 3, k_1 = 2$, and no five points are coplanar;

(c) $n = 3, k_3 = 2$, and a 2-plane β contains exactly six points, say $P_1, \dots, P_6$, such that the planar scheme $V = 2P_1 + \dots + 2P_6 \subset \beta \cong \mathbb{P}^2$ satisfies $\text{reg}(V) = 5$.

2) In all remaining cases, $\text{reg}(Z) = T$.

Proof.

We consider the following cases for the point configuration:

i) The case where there exist $n + 1$ points lying on a linear $n - 1$ -space.

Let

$$k_1 = \min\{t \mid \exists t + 2 \text{ points lying on a linear } t - \text{space}\}.$$

$$\text{We have } T_{k_1} = \left\lfloor \frac{((k_1+2) \cdot 2 + k_1 - 2)}{k_1} \right\rfloor = \left\lfloor 3 + \frac{2}{k_1} \right\rfloor.$$

- For $n = 2$, we only have the case $k_1 = 1$. Then $T_1 = 5, T_2 = 6$. Since the considered point configuration does not lie on two lines, we can assume that P_1, P_2, P_3 are 3 points lying on the line ℓ_1 , P_4, P_5 lie on the line ℓ_2 , and P_6 does not lie on ℓ_1 or ℓ_2 . Let $\wp_i$ be the homogeneous prime ideal of the ring $R = K[X_0, X_1, X_2]$ defined by the point P_i for $i = 1, \dots, 6$. Choosing $P_6 = (1, 0, 0)$, we have $\wp_6 = (X_1, X_2)$. Set $J = \wp_1^2 \cap \dots \cap \wp_5^2$ and $I = J \cap \wp_6^2$. Then R/I is the homogeneous coordinate ring of $Z = 2P_1 + \dots + 2P_6$. Therefore, $\text{reg}(Z) = \text{reg}(R/I)$. By Lemma 2.3, we have: $\text{reg}(R/I) = \max\{2 - 1, \text{reg}(R/J), \text{reg}(R/(J + \wp_6^2))\}$.

Since ℓ_1 passes through the 3 points P_1, P_2, P_3 , we have $\ell_1^2 \in \wp_1^2 \cap \wp_2^2 \cap \wp_3^2$. Since ℓ_2 passes through P_4, P_5 , we have $\ell_2^2 \in \wp_4^2 \cap \wp_5^2$. This implies that $\ell_1^2 \ell_2^2 \in J$. It follows that for any monomial M of degree i in X_1, X_2 ($i = 0, 1$), we have: $\ell_1^2 \ell_2^2 M \in J$.

Since ℓ_j does not pass through P_6 , we can write $\ell_j = X^0 + g_j$ with $g_j \in \wp_6 = (X_1, X_2)$ for $j = 1, 2$. Thus, we have:

$$(X_0 + g_1)^2(X_0 + g_2)^2M \in J.$$

Furthermore, since $g_1, g_2 \in \wp_6$ and $M \in \wp_6^i$, we obtain $X_0^4M \in J + \wp_6^{i+1}$ for $i = 0, 1$. This implies that: $X_0^{5-i}M \in J + \wp_6^{i+1}$ for $i = 0, 1$. By Lemma 2.4, we have:

$$\text{reg}(R/(J + \wp_6^2)) \leq 5.$$

Let $U = 2P_1 + \dots + 2P_5$. We have $T_1(U) = 5 = T_2(U)$, so $T(U) = T_1(U) = 5$. By Lemma 3.1, we have $\text{reg}(U) = T(U) = 5$. Therefore $\text{reg}(R/J) = \text{reg}(U) = 5$. Thus

$$\begin{aligned} \text{reg}(Z) &= \text{reg}(R/I) \\ &= \max\{1, \text{reg}(R/J), \text{reg}(R/(J + \wp_6^2))\} \\ &= 5 = T - 1. \end{aligned}$$

- For $n = 3$, there are 2 cases for k_1 :

+ If $k_1 = 1$, then $T_1 = 5, T_2 = 4, T_3 = 5$. In this case, $T = T_1$, so by Lemma 3.1, we have $\text{reg}(Z) = T_1 = T$.

+ If $k_1 = 2$, then $T_1 = 3, T_2 = 4, T_3 = 5 = T$. Assume first that no five points are coplanar. If five points are coplanar, then $k_2 = 2$, and this subcase is treated in Case ii) below. Let $\wp_j$ be the homogeneous prime ideal of the ring $R = K[X_0, X_1, X_2, X_3]$ defined by the points P_j for $j = 1, \dots, 7$. Set $J = \wp_1^2 \cap \dots \cap \wp_6^2$ and $I = J \cap \wp_7^2$. Then R/I is the homogeneous coordinate ring of Z , hence $\text{reg}(Z) = \text{reg}(R/I)$. By Lemma 2.3, we have:

$$\text{reg}(R/I) = \max\{1, \text{reg}(R/J), \text{reg}(R/(J + \wp_7^2))\}.$$

Assume there exists a linear 2-space α containing 4 points $P_1, \dots, P_4$ and a linear 2-space β containing 3 points P_5, P_6 , and P_7 . Choosing the projective frame $P_1 = (0, 1, 0, 0)$, $P_5 = (0, 0, 1, 0)$, $P_6 = (0, 0, 0, 1)$, $P_7 = (1, 0, 0, 0)$, we have $\wp_7 = (X_1, X_2, X_3)$. Consider the monomial $M = X_1^{c_1}X_2^{c_2}X_3^{c_3}$, with $c_1 + c_2 + c_3 = i$ for $i = 0, 1$.

- Case $|\alpha \cap \beta \cap X| = 1$: Suppose P_4 lies on both α and β .

+ If $i = c_1 = 1$, i.e., $M = X_1$, then M is a hyperplane passing through P_5 and P_6 . We choose the hyperplanes avoiding P_7 as follows: α passing through P_1, P_2, P_3, P_4 ; α_1 passing through P_1, P_5, P_6 . Here, α clearly does not contain P_7 . If α_1 contained P_7 , then since α_1 contains three non-collinear points P_5, P_6, P_7 of β , α_1 would coincide with β and contain 5 points P_1, P_4, P_5, P_6, P_7 , which contradicts our assumption that there are at most $t + 2$ points lying on a linear t -space (with $t = 2$). To ensure that $\alpha^2\alpha_1M \in J$, we check the vanishing multiplicities at each point: since α contains $P_1, \dots, P_4$, we have $\alpha^2 \in \wp_1^2 \cap \dots \cap \wp_4^2$; since α_1 and M both contain P_5 and P_6 , their product α_1M provides a multiplicity of at least 2 at P_5 and P_6 . In this case, $\alpha^2\alpha_1M \in J$.

+ If $i = c_2 = 1$, i.e., $M = X_2$, then M is a hyperplane passing through P_1 and P_6 . We choose the hyperplanes avoiding P_7 as follows: α passing through

P_1, P_2, P_3, P_4 ; α_1 passing through P_2, P_5, P_6 ; and α_2 passing through P_3, P_4, P_5 . Here, α clearly avoids P_7 . If α_1 (or α_2) contained P_7 , it would coincide with β since they share three non-collinear points P_5, P_6, P_7 (or P_4, P_5, P_7 , respectively), leading to 5 points on β , which contradicts $k_1 = 2$. To ensure $\alpha\alpha_1\alpha_2M \in J$, we verify that each point $P_1, \dots, P_6$ is contained in at least two of these hyperplanes. In this case, $\alpha\alpha_1\alpha_2M \in J$.

+ If $i = c_3 = 1$, i.e., $M = X_3$, then M is a hyperplane passing through P_1 and P_5 . We choose the hyperplanes avoiding P_7 as follows: α passing through P_1, P_2, P_3, P_4 ; α_1 passing through P_2, P_5, P_6 ; and α_2 passing through P_3, P_4, P_6 . Here, α avoids P_7 ; if α_1 or α_2 contained P_7 , it would coincide with β and yield five coplanar points, a contradiction. We verify that every point $P_1, \dots, P_6$ lies on at least two of these hyperplanes, counting M . Hence, $\alpha\alpha_1\alpha_2M \in J$.

+ If $i = 0$, i.e., $M = 1$, we choose the hyperplanes avoiding P_7 as follows: α passing through P_1, P_2, P_3, P_4 ; α_1 passing through P_1, P_5, P_6 ; α_2 passing through P_3, P_4, P_6 ; and α_3 passing through P_2, P_4, P_5 . Similarly, if any α_j (for $j = 1, 2, 3$) contained P_7 , it would coincide with β , contradicting $k_1 = 2$. We also verify that each point $P_1, \dots, P_6$ is contained in at least two of these hyperplanes to guarantee $\alpha\alpha_1\alpha_2\alpha_3M \in J$.

- Case $|\alpha \cap \beta \cap X| = 0$:

+ If $i = 0$, i.e., $M = 1$: Let ℓ be the line passing through P_5 and P_6 , and α_1 be a linear 2-space containing ℓ that does not pass through P_7 . Clearly, α avoids P_7 . If the plane α_1 contained P_7 , then since it contains ℓ (which passes through P_5 and P_6), α_1 would contain three non-collinear points P_5, P_6, P_7 of β , meaning α_1 would coincide with β and contain 5 points of X , which contradicts $k_1 = 2$. Furthermore, since α contains $P_1, \dots, P_4$ and α_1 contains P_5, P_6 , the product $\alpha^2\alpha_1^2M$ provides a multiplicity of at least 2 at each point $P_1, \dots, P_6$ to guarantee $\alpha^2\alpha_1^2M \in J$.

+ If $i = c_1 = 1$, i.e., $M = X_1$, then M is a hyperplane passing through P_5 and P_6 . We choose the hyperplanes avoiding P_7 as follows: α passing through P_1, P_2, P_3, P_4 ; α_1 passing through P_1, P_5, P_6 . Note that α_1 does not contain P_7 , because if α_1 contained P_7 , then α_1 would coincide with β since they share three non-collinear points P_5, P_6, P_7 , sharing the point P_1 with α , which contradicts the case under consideration (where $|\alpha \cap \beta \cap X| = 0$). Since α^2 contains $P_1, \dots, P_4$ with multiplicity 2, and α_1M provides a multiplicity of at least 2 at P_5 and P_6 , we guarantee $\alpha^2\alpha_1M \in J$.

+ If $i = c_2 = 1$, i.e., $M = X_2$, then M is a hyperplane passing through P_1 and P_6 . After relabeling P_2, P_3, P_4 , we may assume that the planes through P_3, P_4, P_5 and through P_3, P_4, P_6 avoid P_7 ; otherwise five points would be coplanar. We choose α passing through P_1, P_2, P_3, P_4 ; α_1 passing through P_2, P_5, P_6 ; and α_2 passing through P_3, P_4, P_5 . By this choice, α_2 avoids P_7 ; moreover, α_1 avoids P_7 , since otherwise $\alpha_1 = \beta$, contradicting

$|\alpha \cap \beta \cap X| = 0$. We also verify that each point $P_1, \dots, P_6$ belongs to at least two of these hyperplanes to guarantee $\alpha\alpha_1\alpha_2M \in J$.

+ If $i = c_3 = 1$, i.e., $M = X_3$, then M is a hyperplane passing through P_1 and P_5 . With the same relabeling, we choose α passing through P_1, P_2, P_3, P_4 ; α_1 passing through P_2, P_5, P_6 ; and α_2 passing through P_3, P_4, P_6 . Again, α_2 avoids P_7 , and α_1 avoids P_7 since otherwise $\alpha_1 = \beta$, contradicting $|\alpha \cap \beta \cap X| = 0$. We also verify that each point $P_1, \dots, P_6$ belongs to at least two of these hyperplanes to guarantee $\alpha\alpha_1\alpha_2M \in J$.

By Lemma 2.4, we have $\text{reg}\left(\frac{R}{J+\wp^2}\right) \leq 4$. Let $U = 2P_1 + \dots + 2P_6$. We have $T_1(U) = 3$, $T_2(U) = 4$, $T_3(U) = 4$, so $T(U) = T_2(U) = 4$. Since we have exactly 6 double points ($m_1 = \dots = m_6 = 2$) in the 3-dimensional space, the parameters do not satisfy the condition of the exceptional case of Lemma 2.5 (which strictly requires five points of multiplicity m and one point of multiplicity $m - 1$). Thus, this configuration does not fall into the exceptional case. By Lemma 2.5, we obtain $\text{reg}(U) = T(U) = 4$. Since $\text{reg}(R/J) = \text{reg}(U)$, we have $\text{reg}(R/J) = 4$. Thus, in this case:

$$\text{reg}(Z) = \text{reg}(R/I) = \max\{1, \text{reg}(R/J), \text{reg}(R/(J + \wp^2))\} = 4 = T - 1.$$

ii) Case where there exist $n + 2$ points lying on a linear $n - 1$ -space. Let

$$k_2 = \min\{t \mid \exists t + 3 \text{ points lying on a linear } t\text{-space}\}.$$

$$\text{We have } T_{k_2} = \left\lceil \frac{((k_2+3) \cdot 2 + k_2 - 2)}{k_2} \right\rceil = \left\lceil 3 + \frac{4}{k_2} \right\rceil.$$

For $n = 2$, we only have the case $k_2 = 1$. Then $T_1 = 7, T_2 = 6, T = \max\{T_1, T_2\} = T_1$. By Lemma 3.1, we have $\text{reg}(Z) = T_1 = T$.

For $n = 3$, there are 2 cases for k_2 :

+ If $k_2 = 1$, then $T_1 = 7, T_2 = 5, T_3 = 5$. In this case, $T = T_1$, so by Lemma 3.1, we have $\text{reg}(Z) = T_1 = T$.

+ If $k_2 = 2$, assume first that no six points are coplanar; the remaining subcase is treated in Case iii). Then $T_1 \leq 5, T_2 = 5, T_3 = 5$. Thus $T = T_2$. Let α be a 2-flat passing through 5 points $P_{i_1}, \dots, P_{i_5}$ such that $T(\alpha) = T_2 = 5$. Consider the fat point scheme $U = 2P_{i_1} + \dots + 2P_{i_5}$. The five points $P_{i_1}, \dots, P_{i_5}$ lie on a conic in α . Hence, the square of an equation of this conic is a nonzero quartic vanishing doubly at these five points. Since the vector space of quartics on $\alpha \cong \mathbb{P}^2$ has dimension 15, which is the length of U , we have $H_U(4) < 15$, so $\text{reg}(U) \geq 5$. By Lemma 2.2, $\text{reg}(Z) \geq \text{reg}(U) \geq 5$, while Lemma 2.6 gives $\text{reg}(Z) \leq T = 5$. Therefore, $\text{reg}(Z) = T = 5$.

iii) Case where there exist $n + 3$ points lying on an $(n - 1)$ -flat. Let

$$k_3 = \min\{t \mid \exists t + 4 \text{ points lying on a linear } t\text{-space}\}.$$

$$\text{We have } T_{k_3} = \left\lceil \frac{((k_3+4) \cdot 2 + k_3 - 2)}{k_3} \right\rceil = \left\lceil 3 + \frac{6}{k_3} \right\rceil.$$

For $n = 2$, we only have the case $k_3 = 1$. Then

$T_1 = 9, T_2 = 6, T = \max\{T_1, T_2\} = T_1$. By Lemma 3.1, we have $\text{reg}(Z) = T_1 = T$.

For $n = 3$, there are 2 cases for k_3 :

+ If $k_3 = 1$, then $T_1 = 9, T_2 = 6, T_3 = 5$. In this case, $T = T_1$, so by Lemma 3.1, we have $\text{reg}(Z) = T_1 = T$.

+ If $k_3 = 2$, then $T_1 \leq 7, T_2 = 6, T_3 = 5$. Thus $T = T_1$ or $T = T_2$. If $T = T_1$, then by Lemma 3.1, we have $\text{reg}(Z) = T$. If $T = T_2$, without loss of generality or by reindexing, we consider β to be a 2-flat in $\mathbb{P}^3$ passing through 6 points $P_1, \dots, P_6$ such that $T(\beta) = T_2 = 6$. Consider the fat point scheme $V = 2P_1 + \dots + 2P_6$.

The space of quartics on $\beta \cong \mathbb{P}^2$ has dimension $15 < 18$, the length of V ; hence $\text{reg}(V) \geq 5$, while Lemma 2.6 gives $\text{reg}(V) \leq 6$. If $\text{reg}(V) = 6$, then Lemmas 2.2 and 2.6 give $6 = \text{reg}(V) \leq \text{reg}(Z) \leq T = 6$, so $\text{reg}(Z) = T$. Assume now that $\text{reg}(V) = 5$. Let the point $P_7 = (1, 0, 0, 0) \in \mathbb{P}^3$. Let $\wp_i$ be the homogeneous prime ideal defined by P_i for $i = 1, \dots, 7, J = \wp_1^2 \cap \dots \cap \wp_6^2$, $I = J \cap \wp_7^2$, and $R = K[X_0, \dots, X_3]$. Then we have: $\text{reg}(Z) = \text{reg}(R/I) = \max\{1, \text{reg}(R/J), \text{reg}(R/(J + \wp_7^2))\}$. Moreover, $\text{reg}(R/J) = \text{reg}(V) = T - 1 = 5$.

Since $P_7 \notin \beta$ and $P_1, \dots, P_6 \in \beta$, we have $\beta^2 \in \wp_1^2 \cap \dots \cap \wp_6^2 = J$. This implies that $\beta^{4-i}M \in J$ for any monomial M of degree i in X_1, X_2, X_3 ($i = 0, 1$). Since $P_7 \notin \beta$, we can write $\beta = X_0 + g$, where $g \in \wp_7 = (X_1, X_2, X_3)$ is a linear form. Therefore, we have $(X_0 + g)^{4-i}M \in J$ for any monomial M of degree i in X_1, X_2, X_3 ($i = 0, 1$). Furthermore, since $g \in \wp_7$ and $M \in \wp_7^i$, we obtain $X_0^{4-i}M \in J + \wp_7^{i+1}$ for any monomial M of degree i in X_1, X_2, X_3 ($i = 0, 1$).

By Lemma 2.4, we have $\text{reg}(R/(J + \wp_7^2)) \leq 4$. Therefore, $\text{reg}(Z) = 5 = T - 1$.

Theorem 3.2 is proved.

Remark 3.3. We emphasize the importance of the hypothesis that the points do not lie on two lines. Determining the exact regularity index for fat points on two lines is an extremely difficult problem that remains completely open and unsolved by any author to date. We deliberately exclude this highly challenging case to maintain a feasible scope.

4. Conclusion

This paper completely determines the explicit formulas for the regularity index of a zero-dimensional scheme consisting of $n + 4$ non-degenerate double points not lying on two lines in the projective space $\mathbb{P}^n$ for $n = 2$ and $n = 3$. By partitioning the configurations based on the geometric quantities T_j and k_j , we have successfully provided precise, explicitly computed values for each distinct subcase. Settling these base dimensions yields crucial geometric insights and provides an essential baseline for general structures.

Extending these results to $\mathbb{P}^n$ for $n \geq 4$ presents significant challenges, primarily due to the combinatorial explosion of point configurations. This complexity makes

classifying geometric invariants much more difficult, preventing direct applications of standard induction and hyperplane restriction techniques. Future work could leverage advanced combinatorial methods or computational algorithms to overcome these dimensional barriers.

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