SURFACES IN $\mathbb{P}^4$ DEFINED BY THREE-POINT SUBSCHEMES OF LENGTH 5

MẶT TRONG $\mathbb{P}^4$ ĐỊNH BỞI CÁC LƯỢC ĐỒ BA ĐIỂM CÓ ĐỘ DÀI 5

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Abstract - Studying algebraic varieties through linear systems of homogeneous forms with prescribed base loci provides valuable insights into their geometric properties and invariants. It is well known that cubic forms in $\mathbb{P}^2$ with six base points in general position, define smooth cubic surfaces with 27 lines in $\mathbb{P}^3$. Recent works have focused on linear systems with four or five base points, whose images are surfaces of degree 3 or 4 in $\mathbb{P}^4$. In this paper, we investigate linear systems of cubic forms defined by a three-point subscheme of length 5 involving fat points. We analyze the resulting configurations, including their degrees, singular loci, and the number of lines on the associated surfaces, thereby contributing to the classification of surfaces in $\mathbb{P}^4$ and clarifying the correspondence between linear systems and varieties, with potential applications in information technology and related engineering fields.

Key words - Linear systems; algebraic varieties; fat points; subschemes; geometric invariants

Tóm tắt - Việc nghiên cứu các đa tạp thông qua các hệ tuyến tính của các dạng thuần nhất với tập cơ sở cho trước có nhiều lợi thế trong việc khám phá các tính chất và các bất biến của chúng. Chúng ta biết rằng, hệ tuyến tính các dạng bậc ba trong $\mathbb{P}^2$ với sáu điểm ở vị trí tổng quát xác định một mặt bậc ba trơn chứa 27 đường thẳng trong $\mathbb{P}^3$. Gần đây, nhiều kết quả phân loại đã được thiết lập cho các hệ tuyến tính với bốn hoặc năm điểm cơ sở, mà đa tạp tương ứng là các mặt bậc 3 hoặc 4 trong $\mathbb{P}^4$. Trong bài báo này, chúng tôi nghiên cứu các hệ tuyến tính các dạng bậc ba xác định bởi một lược đồ ba điểm có độ dài 5 chứa các điểm bội. Chúng tôi phân tích bậc, tập kỳ dị và số lượng các đường thẳng của các mặt thu được, qua đó góp phần phân loại các mặt trong $\mathbb{P}^4$, với những tiềm năng ứng dụng trong công nghệ thông tin và các lĩnh vực kỹ thuật liên quan.

Từ khóa - Hệ tuyến tính; đa tạp; điểm béo; lược đồ; bất biến hình học

1. Introduction

Throughout this paper, we work over an algebraically closed field k. Let $Z = m_1P_1 + \cdots + m_rP_r$ be a zero-dimensional subscheme supported on r distinct points in the projective plane $\mathbb{P}^2 = \mathbb{P}^2_k$, with length $m = \sum_{i=1}^{r} m_i$. Let $\mathcal{L}_Z$ be the linear subspace of the space of homogeneous forms of the same degree associated with Z, and let $\mathrm{Supp}(\mathcal{L}_Z) \subset \mathbb{P}^2$ be the variety defined by $\mathcal{L}_Z$. Assuming that $\dim(\mathcal{L}_Z) = n$, we let $\{F_1, \dots, F_n\}$ be a basis of $\mathcal{L}_Z$. Consider the rational map:

$$\begin{array}{rcl} \varphi\colon \mathbb{P}^2\backslash \mathrm{Supp}(\mathcal{L}_Z) & \to & \mathbb{P}^{n-1} \\ P & \mapsto & \big(F_1(P):\dots:F_n(P)\big). \end{array} \tag{1.1}$$

Let $X = \overline{\mathrm{Im}(\varphi)}$ be the Zariski closure of the image of φ in $\mathbb{P}^{n-1}$.

Lemma 1. *The variety X defined in (1.1) does not depend on the choice of a basis. Furthermore, if φ is injective on an open subset, then X is a surface whose degree is equal to the number of intersection points of two generic curves in $\mathcal{L}_Z$ outside* $\mathrm{Supp}(\mathcal{L}_Z)$.

Proof. Let $\{G_1, \dots, G_n\}$ be another basis of $\mathcal{L}_Z$, and let X' be the corresponding Zariski closure defined via (1.1) with respect to this new basis. Let $M \in \mathrm{GL}(n)$ be the transition matrix from $\{F_1, \dots, F_n\}$ to $\{G_1, \dots, G_n\}$. Then M induces a projective automorphism of $\mathbb{P}^{n-1}$ that maps X onto X', which proves that the definition of X is independent of the choice of a basis up to projective equivalence.

For the second assertion, let C_1 and C_2 be two generic forms in $\mathcal{L}_Z$. Their coordinates with respect to a given basis correspond to two hyperplanes H_1 and H_2 in $\mathbb{P}^{n-1}$. The intersection points of X with $H_1 \cap H_2$ are in one-to-one correspondence with the intersection points of $C_1 \cap C_2$ away from $\mathrm{Supp}(\mathcal{L}_Z)$. By definition, the number of these intersection points constitutes the degree of the variety X.

By virtue of Lemma 1, X is called the variety associated with $\mathcal{L}_Z$. Studying the variety X through the linear system $\mathcal{L}_Z$ offers significant advantages. It facilitates not only the understanding of crucial geometric properties-such as irreducibility, singularities, and the complete intersection property-but also the accurate evaluation of the dimension and degree of X. Furthermore, it allows one to determine the number and configuration of both its singularities and lines, alongside other quantitative invariants of X (see [1, 2]).

It is well-known that the linear system of cubic forms containing six points in general position corresponds to a smooth cubic surface with 27 lines in $\mathbb{P}^3$ (see [3 - 8]). In recent years, further research on linear systems of cubic forms through six points not in general position has yielded interesting results corresponding to singular and semi-stable cubic surfaces ([8 - 11]). Recently, several results have been obtained for the cases of linear systems of cubic forms whose base loci consist of four or five distinct points (see [12, 13]). All the corresponding varieties are surfaces in $\mathbb{P}^4$ endowed with explicit data regarding their degrees, number of lines, and singular points.

This paper extends the aforementioned results to linear systems with a base locus consisting of exactly three distinct points, such that the projective dimension of the linear system remains 4. In these cases, the base locus contains one or two fat points and maintains a total length of 5. Specifically, we focus on investigating the following geometric configurations:

- *Three points in general position with two prescribed tangent lines at two base points*: The linear system consists of all cubic forms passing through the three points and tangent to given lines d_1, d_2 at two of the designated points. This configuration further admits five specializations when the tangent lines pass through other base points.

- *Three collinear points with two prescribed tangent lines at two base points*: The linear system consists of all cubic forms passing through the three collinear points and tangent to given lines d_1, d_2 at two of the designated points. This configuration further admits a specialization when one of the tangent lines passes through other base points.

- *Three points in general position, one of which is a fat point of multiplicity 3*: The linear system consists of all cubic forms passing through the three base points and having a singular point of multiplicity 3 at one of the designated points.

- *Three collinear points, one of which is a fat point of multiplicity 3*: The linear system consists of all cubic forms passing through the three collinear points and having a singular point of multiplicity 3 at one of the designated points.

2. Surfaces defined by linear systems of cubics having 3 base points of length 5

Theorem 1. *Let $Z = 2P_1 + 2P_2 + P_3$ be a subscheme in $\mathbb{P}^2$, where the three support points are non-collinear. Let $\mathcal{L}_Z$ be the linear system of cubic forms passing through the points, tangent to a given line d_1 at P_1, and tangent to another line d_2 at P_2. Then the variety X defined by $\mathcal{L}_Z$ is a surface of degree 4 in $\mathbb{P}^4$ having two singular points and six lines if d_1 or d_2 does not contain any other base points. Furthermore, there are the following specializations where the varieties defined by $\mathcal{L}_Z$ are all surfaces in $\mathbb{P}^4$:*

1. If exactly one of d_1, d_2 contains P_3, then the surface is of degree four and has three singular points and three lines.

2. If both d_1, d_2 contain P_3, then the surface is of degree four and has four singular points and one line.

3. If d_i contains P_j for $i \neq j \in \{1,2\}$, and d_j does not contain any other base points, then the surface is of degree four and has two singular points and three lines.

4. If d_i contains P_j for $i \neq j \in \{1,2\}$, and d_j contains P_3, then the surface is of degree 4 and has three singular points and one line.

5. If $d_1 = d_2 = \overline{P_1P_2}$, then the surface X is of degree three and decomposes into a 2-plane and a surface of degree two with infinitely many lines; the singular locus of the surface X consists of two projective lines.

Proof 1. Except for the special case where $d_1 = d_2 = \overline{P_1P_2}$, the support of the linear system $\mathrm{Supp}(\mathcal{L}_Z) = \{P_1, P_2, P_3\}$. We see that $\mathcal{L}_Z$ is a vector space of dimension 5 and the rational map φ defined as in (1.1) is injective at least outside of $d_1 \cup d_2$ and the base points. Under these specific tangential constraints, two generic cubics in $\mathcal{L}_Z$ share exactly 4 common points outside $\mathrm{Supp}(\mathcal{L}_Z)$, which implies that X is a surface of degree 4 in $\mathbb{P}^4$.

If the line d_1 or d_2 does not contain other base points, then the lines $\overline{P_1P_2}$, $\overline{P_1P_3}$, and $\overline{P_2P_3}$ intersect any cubic curve in $\mathcal{L}_Z$ at one additional point. This implies that their images yield three lines on the surface X. With the same argument, the images of d_1, d_2 and of the conic curve C containing all the base points and tangent to d_1, d_2 at P_1, P_2 respectively, constitute three other lines on X. Conversely, if d_1 or d_2 passes through P_3, the image collapses to a singular point on X, and hence the surface contains fewer lines.

Without loss of generality, we can assume that $P_1 = (1{:}0{:}0)$, $P_2 = (0{:}1{:}0)$, and $P_3 = (0{:}0{:}1)$ in $\mathbb{P}^2$ with coordinates $(u{:}v{:}w)$. Let $d_1 = V(av + bw)$ and $d_2 = V(cu + dw)$ be two tangent lines at P_1 and P_2 respectively, for some $a, b, c, d \in k$. A direct computation shows that a basis for the linear system $\mathcal{L}_Z$ consists of the following five cubic forms:

$$f_1 = au^2v + bu^2w, \quad f_2 = uvw, \quad f_3 = uw^2,$$
$$f_4 = cuv^2 + dv^2w, \quad f_5 = vw^2,$$

where $b, d \neq 0$. Consequently, the surface $X \subset \mathbb{P}^4$ with homogeneous coordinates $(x{:}y{:}z{:}t{:}s)$ is defined by two polynomials:

$$g_1 = cy^2 + dys - dzt, \quad g_2 = bxs - ay^2 - bzy.$$

Then X has two singular points $(1{:}0{:}0{:}0{:}0)$ and $(0{:}0{:}0{:}1{:}0)$.

We now analyze the specific degeneration cases:

1. In the first scenario, suppose that d_1 contains $P_3 = (0{:}0{:}1)$, where the corresponding surface is defined by:

$$g_1 = xs - y^2, \quad g_2 = -cxs - dys + dzt,$$

where $a, d \neq 0$.

Under this geometric setting, the image of $d_1 = \overline{P_1P_3}$ collapses to an additional singular point $(0{:}0{:}1{:}0{:}0)$, thereby yielding a configuration of three singular points and three lines on the surface.

2. When both tangent lines d_1 and d_2 pass through P_3, the equations defining the surface reduce to:

$$g_1 = zt - xs, \quad g_2 = y^2 - xs.$$

In this case, the image of $d_2 = \overline{P_2P_3}$ collapses to another singular point $(0{:}0{:}0{:}0{:}1)$, which consequently restricts the surface to containing a unique line.

3. Alternatively, consider the case where $d_1 = \overline{P_1P_2}$. The image of d_1 then collapses to the singular point $(1{:}0{:}0{:}0{:}0)$, and the surface X is defined by:

$$g_1 = -y^2 - ys + zt, \quad g_2 = -xs + yz.$$

Provided that d_2 does not pass through any other base points, the surface preserves two singular points and contains three lines.

3. Under the further joint assumption that $d_1 = \overline{P_1P_2}$ and d_2 contains P_3, the image of d_2 collapses to another singular point $(0:0:0:0:1)$. Here, the surface X is explicitly defined by:

$$g_1 = -xt + y^2, \quad g_2 = -xy + zs,$$

exhibiting three singular points and a unique line.

4. Finally, we address the maximal degeneration where $d_1 = d_2 = \overline{P_1P_2}$. The corresponding surface X is defined by:

$$g_1 = yz - xs, \quad g_2 = y^2 - xt.$$

As a consequence, the variety decomposes into two components $X = X_1 \cup X_2$, where $X_1 = V(x, y)$ is a 2-plane and $X_2 = V(yz - xs, y^2 - xt, sy - zt)$ is a surface of degree two. The singular locus of X consists of two projective lines:

$$L_1 = V(x, y, z) \quad \text{and} \quad L_2 = V(x, y, t),$$

which intersect transversely at a single point $(0:0:1:0:0)$ □

Theorem 2. *Let $Z = 2P_1 + 2P_2 + P_3$ be a subscheme in $\mathbb{P}^2$, where the support points are collinear. Let $\mathcal{L}_Z$ be the linear system of cubic forms passing through P_1, P_2, P_3 and tangent to a given line d_1 at P_1, tangent to another line d_2 at P_2. Then the variety X defined by $\mathcal{L}_Z$ is a surface of degree 4 in $\mathbb{P}^4$ with one singular point and containing two lines if d_1 or d_2 does not contains P_3.*

In the case, exactly one of d_1, d_2 passes through P_3, then the surface has degree three and decomposes into a 2-plane and a surface of degree two with infinitely many lines; the singular locus of the surface X consists of two projective lines.

Proof 2. Let l be the line connecting P_1, P_2, P_3. If d_1 or d_2 does not containsP_3 then the support of the linear system $\mathrm{Supp}(\mathcal{L}_Z) = \{P_1, P_2, P_3\}$. We see that $\mathcal{L}_Z$ is a vector space of dimension 5, and the rational map φ defined as in (1.1) is injective at least outside of $d_1 \cup d_2$ and the base points. Since two generic cubics in $\mathcal{L}_Z$ share 4 common points outside $\mathrm{Supp}(\mathcal{L}_Z)$, it follows that X is a surface of degree 4. The image of l is a singular point on X.

Moreover, the lines d_1 and d_2 intersect any cubic curve in $\mathcal{L}_Z$ at on additional point. This implies that their images yield two lines on the surface X.

Without loss of generality, we can assume that $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (1:1:0)$ in $\mathbb{P}^2$ with coordinates $(u:v:w)$. Let $d_1 = V(av + bw)$ and $d_2 = V(cu + dw)$ be two tangent lines at P_1 and P_2 respectively for some $a, b, c, d \in k$. A direct computation shows that a basis for $\mathcal{L}_Z$ consists of the following five cubic forms:

$$f_1 = vw^2,$$
$$f_2 = -acu^2v - bcu^2w + cauv^2 + adv^2w,$$
$$f_3 = uw^2, \; f_4 = uvw, \; f_5 = w^3.$$

Consequently, the surface $X \subset \mathbb{P}^4$ with homogeneous coordinates $(x:y:z:t:s)$ is defined by two polynomials:

$$g_1 = -adx^2 - acxt + adys + bcz^2 + aczt,$$
$$g_2 = -xz + ts.$$

Then X has one singular point $(0:1:0:0:0)$, that is the image of l.

Consider now the case where $d_1 = l$ and $d_2 \neq l$. Then each curve in $\mathcal{L}_Z$ factors into the line l and a conic containing P_1. The support $\mathrm{Supp}(L_Z) = l \cup \{P_1, P_2, P_3\}$. Two generic forms in $\mathcal{L}_Z$ share 3 commons points outside $\mathrm{Supp}(\mathcal{L}_Z)$, hence the corresponding surface X has degree 3. We can choose a basis of L_Z for this case:

$$f_1 = vw^2, \; f_2 = u^2w, \; f_3 = uw^2,$$
$$f_4 = uvw, \; f_5 = w^3.$$

The surface X is defined by

$$g_1 = -ys + z^2, \; g_2 = -xz + ts,$$

and decomposes into two components $X = X_1 \cup X_2$, where $X_1 = V(z, s)$ is a 2-plane and $X_2 = V(-ys + z^2, -xz + st, xy - zt)$ is a surface of degree two. The singular locus of X consists of two projective lines:

$$L_1 = V(x, z, s) \quad \text{and} \quad L_2 = V(y, z, s),$$

which intersect transversely at a single point $(1:0:0:0:0)$. □

Theorem 3. *Let $Z = 3P_1 + P_2 + P_3$ be a subscheme in $\mathbb{P}^2$, where the support points are non-collinear. Let $\mathcal{L}_Z$ be the linear system of cubic forms passing through these points and being singular at P_1. Then the variety X defined by $\mathcal{L}_Z$ is a surface in $\mathbb{P}^4$ of degree three, which decomposes into a 2-plane and a surface of degree two with infinitely many lines; the singular locus of the surface X consists of two projective lines.*

Proof 3. We see that $\mathrm{Supp}(\mathcal{L}_Z) = \{P_1, P_2, P_3\}$, the vector space $\mathcal{L}_Z$ is a vector space of dimension 5, and the rational map φ defined as in (1.1) is injective outside the base points. Since two generic cubics in $\mathcal{L}_Z$ share 3 common points outside $\mathrm{Supp}(\mathcal{L}_Z)$, it follows that X is a surface of degree 3.

Without loss of generality, we can assume that $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, and $P_3 = (0:0:1)$ in $\mathbb{P}^2$ with coordinates $(u:v:w)$. Then the linear system $\mathcal{L}_Z$ consists of the following five cubic forms:

$$f_1 = uv^2, \quad f_2 = uw^2, \quad f_3 = uvw,$$
$$f_4 = vw^2, \quad f_5 = v^2w.$$

The surface $X \subset \mathbb{P}^4$ with homogeneous coordinates $(x:y:z:t:s)$ is defined by two polynomials:

$$g_1 = -ys + zt, \quad g_2 = -xy + z^2.$$

Then the surface decomposes into two components $X = X_1 \cup X_2$, where $X_1 = V(y, z)$ is a 2-plane and $X_2 = V(zt - ys, z^2 - xy, xt - zs)$ is a surface of degree two. The singular locus of X consists of two projective lines:

$$L_1 = V(x, y, z) \quad \text{and} \quad L_2 = V(y, z, t),$$

which intersect transversely at a single point $(0:0:0:1:0)$.□

Theorem 4. *Let $Z = 3P_1 + P_2 + P_3$ be a subscheme in $\mathbb{P}^2$, where the support points are collinear. Let $\mathcal{L}_Z$ be the linear system of cubic forms passing through these points and being singular at P_1. Then the variety X defined by $\mathcal{L}_Z$ is a surface in $\mathbb{P}^4$ of degree three, which decomposes into a 2-plane and a surface of degree two with infinitely many lines; the singular locus of the surface X consists of two projective lines.*

***Proof** 4*. Let l be the line containing P_1, P_2, P_3. Any cubic form in $\mathcal{L}_Z$ factors into the line l and a conic passing through P_1. We see that $\mathcal{L}_Z$ is a vector space of dimension 5 and the $\mathrm{Supp}(\mathcal{L}_Z) = l \cup \{P_1, P_2, P_3\}$; the rational map φ defined as in (1.1) is injective outside $\mathrm{Supp}(\mathcal{L}_Z)$. Since two generic cubics in $\mathcal{L}_Z$ share 3 common points outside $\mathrm{Supp}(\mathcal{L}_Z)$, the corresponding surface X is of degree 3 in $\mathbb{P}^4$. In fact $\mathcal{L}_Z$ is the same as $\mathcal{L}_Z$ in the degeneration case considered in Theorem 2. Therefore, the surface has degree three and decomposes into a 2-plane and a surface of degree two with infinitely many lines; the singular locus of the surface X consists of two projective lines

3. Conclusion

This paper investigated the projective varieties associated with linear systems of cubic forms in $\mathbb{P}^2$ passing through three distinct points under four configurations with several degenerations. In all cases, the corresponding varieties X are embedded as surfaces of degree 3 or 4 in $\mathbb{P}^4$. By examining their underlying linear systems, we have uncovered various profound geometric properties of these surfaces, including their degrees, defining equations, singularities, and the configurations of lines they contain. These results not only deepen the interplay between algebraic geometry and computational algebra but also lay a solid foundation for potential applications in information technology and related engineering fields.

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