ABOUT THE POLYNOMIALS SOLUTIONS OF CONTROL SYSTEMS

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Abstract - In this paper, we propose a method to build up solutions (state functions) of the control systems, which transfers the system from any initial conditions in to any final conditions and at the same time satisfies conditions given to the controllability function *u(t)* which makes it possible to find in the type of polynomials of degree $((p+1)(k+2)-1)$ with vector coefficients. In the final step, we obtain a pseudo-state function $x_p(t)$ satisfying the conditions and substituting this in the previous step. The method is based on the splitting of the spaces into subspaces and the transition from the original equation to the same equation with the subjective matrix.

Key words - control systems; state functions; control functions; polynomial solutions; control points.

1. Statement of the problem

We consider the control system:

$$\frac{dx(t)}{dt} = Bx(t) + Du(t), \tag{1}$$

where $B \in L(R^n, R^n), D \in L(R^m, R^n), x(t) \in R^n$,

$u(t) \in R^m, t \in [0,T]$, with conditions:

$$x(t_i) = \gamma_i, i = \overline{0, k+1}, t_0 = 0, t_{k+1} = T, \tag{2}$$

$(t_j, \gamma_j), j = \overline{1,k}$, are check points.

In the case of a fully controlled[1] system (1) the task of managing the search function $u(t)$ in the polynomial form, which takes the system from any state γ_0 to any state γ_{k+1} for time T, and the trajectory of the system $x(t)$ will pass through the check points $(t_j, \gamma_j), j = \overline{1,k}$.

2. Results

Specify the following theorem:

Theorem. *There exists a control functions $u(t)$ as a polynomial according t, whose order is less than or equal to $((p+1)(k+2)-1)$,*[2] so *that the solution $x(t)$ of problem (1)* - (2) is a polynomial of degree $((p+1)(k+2)-1)$.

To prove the theorem we use the following lemma (see[1]):

Lemma. *Fully manage the system (1)* with conditions $x(0) = \gamma_0, x(T) = \gamma_{k+1}$ *is equivalent to*:

$$\begin{cases} u(t) = D^+ \dfrac{dx(t)}{dt} - D^+ Bx(t) + Pu(t), \\ x_{i-1}(t) = x_i(t) + y_i(t), \\ y_i(t) = D_i^+ \dfrac{dx_i(t)}{dt} - D_i^+ B_i x_i(t) + P_i y_i(t), \\ \dots \\ x_{p-1}(t) = x_p(t) + y_p(t), \\ \dfrac{dx_p(t)}{dt} = B_p x_p(t) + D_p y_p(t) \end{cases} \tag{3}$$

with conditions

$$\left.\frac{d^j x_p}{dt^j}\right|_{t=0} = \gamma^j_{0,p}, \left.\frac{d^j x_p}{dt^j}\right|_{t=T} = \gamma^j_{k+1,p}, j = \overline{0,p}. \tag{4}$$

3. Proof of theorem

With cascade splitting of the system (1) for p steps to move the system (3), at the same time, on each i - th step of splitting get exactly $k+2$ additional conditions at the points $t_i, i = \overline{0, k+1}$ on the function $x_i(t)$ of the state of each step. That is, from the conditions (2), we pass to the equivalent $(p+1)(k+2)$ conditions on the pseudo-states function $x_p(t)$ of the last equation of (3) and its derivatives up to and including p-th order:

$$\begin{aligned} &\left.\frac{d^j x_p(t)}{dt^j}\right|_{t=0} = \gamma^j_{0,p}, \left.\frac{d^j x_p(t)}{dt^j}\right|_{t=t_1} = \gamma^j_{1,p}, \\ &\left.\frac{d^j x_p(t)}{dt^j}\right|_{t=t_2} = \gamma^j_{2,p}, \dots, \left.\frac{d^j x_p(t)}{dt^j}\right|_{t=t_k} = \gamma^j_{k,p}, \\ &\left.\frac{d^j x_p(t)}{dt^j}\right|_{t=t_{k+1}=T} = \gamma^j_{k+1,p}, j = \overline{0,p}. \end{aligned} \tag{5}$$

We seek the function $x_p(t)$ as a polynomial according t of degree $((p+1)(k+2)-1)$:

$$x_p(t) = \sum_{j=0}^{(p+1)(k+2)-1} c_j t^j. \tag{6}$$

Substituting the first condition of system (5) when $j=0$ in the expansion (6), we find the value $c_0 = \gamma^0_{0,p}$. Differentiating the series (6), with the first condition (5) when $j = \overline{1,p}$, we find the values of the first $(p+1)$ coefficients of the expansion (6):

$$c_j = \frac{1}{j!}\gamma^j_{0,p}, j = \overline{1,p}.$$

The substitution of the following conditions (5) in the expression (6) and its derivatives up to and including p-th order, leads to a system with respect to the expansion coefficients $c_j, j = p+1, (p+1)(k+2)-1$ of (6):

$$\begin{cases} t_1^{p+1} c_{p+1} + t_1^{p+2} c_{p+2} + \dots + t_1^{(p+1)(k+2)-1} c_{(p+1)(k+2)-1} = \\ = \gamma^0_{1,p} - \displaystyle\sum_{j=0}^{p} \frac{1}{j!}\gamma^j_{0,p} t_1^j, \\ (p+1)t_1^p c_{p+1} + (p+2)t_1^{p+1} c_{p+2} + \dots + \\ [(p+1)(k+2)-1] t_1^{(p+1)(k+2)-2} c_{(p+1)(k+2)-1} = \\ = \gamma^1_{1,p} - \displaystyle\sum_{j=1}^{p} \frac{1}{(j-1)!}\gamma^j_{0,p} t_1^{j-1}, \\ \dots. \end{cases}$$

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[1] see [7], [8].

[2] $p \in N$ is such that the matrix D_p surjective (see [7], [9]), k - number of check points.

$$\begin{cases} (p+1)p...2t_1c_{p+1}+(p+2)(p+1)...3t_1^2c_{p+2}+ \\ ...+[(p+1)(k+2)-1][(p+1)(k+2)-2].... \\ [(p+1)(k+1)+1]t_1^{(p+1)(k+1)}=\gamma_{1,p}^p-\gamma_{0,p}^p, \\ t_2^{p+1}c_{p+1}+t_2^{p+2}c_{p+2}+...+t_2^{(p+1)(k+2)-1}c_{(p+1)(k+2)-1}= \\ \gamma_{2,p}^0-\sum_{j=0}^{p}\frac{1}{j!}\gamma_{0,p}^j t_2^j, \\ (p+1)t_2^p c_{p+1}+(p+2)t_2^{p+1}c_{p+2}+... \\ +[(p+1)(k+2)-1]t_2^{(p+1)(k+2)-2}c_{(p+1)(k+2)-1}= \\ =\gamma_{2,p}^1-\sum_{j=1}^{p}\frac{1}{(j-1)!}\gamma_{0,p}^j t_2^{j-1}, \\ .... \end{cases}$$

$$\begin{cases} t_k^{p+1}c_{p+1}+t_k^{p+2}c_{p+2}+...+t_k^{(p+1)(k+2)-1}c_{(p+1)(k+2)-1}= \\ \gamma_{k,p}^0-\sum_{j=0}^{p}\frac{1}{j!}\gamma_{0,p}^j t_k^j, \\ (p+1)t_k^p c_{p+1}+(p+2)t_k^{p+1}c_{p+2}+... \\ +[(p+1)(k+2)-1]t_k^{(p+1)(k+2)-2}c_{(p+1)(k+2)-1}= \\ =\gamma_{k,p}^1-\sum_{j=1}^{p}\frac{1}{(j-1)!}\gamma_{0,p}^j t_k^{j-1}, \\ .... \\ (p+1)p...2t_kc_{p+1}+(p+2)(p+1)...3t_k^2c_{p+2}+ \\ ...+[(p+1)(k+2)-1][(p+1)(k+2)-2].... \\ [(p+1)(k+1)+1]t_k^{(p+1)(k+1)}=\gamma_{k,p}^p-\gamma_{0,p}^p, \end{cases}$$

$$\begin{cases} T^{p+1}c_{p+1}+T^{p+2}c_{p+2}+...+T^{(p+1)(k+2)-1}c_{(p+1)(k+2)-1}= \\ \gamma_{k+1,p}^0-\sum_{j=0}^{p}\frac{1}{j!}\gamma_{0,p}^j T^j, \\ (p+1)T^p c_{p+1}+(p+2)T^{p+1}c_{p+2}+...+ \\ [(p+1)(k+2)-1]T^{(p+1)(k+2)-2}c_{(p+1)(k+2)-1}= \\ =\gamma_{k+1,p}^1-\sum_{j=1}^{p}\frac{1}{(j-1)!}\gamma_{0,p}^j T^{j-1}, \\ .... \\ (p+1)p...2Tc_{p+1}+(p+2)(p+1)...3T^2c_{p+2}+... \\ +[(p+1)(k+2)-1][(p+1)(k+2)-2].... \\ [(p+1)(k+1)+1]T^{(p+1)(k+1)}c_{(p+1)(k+2)-1}=\gamma_{k+1,p}^p-\gamma_{0,p}^p, \end{cases} \tag{7}$$

The determinant Δ_1 of system (7) is set to:

$$\Delta_1=(\prod_{i=1}^{k+1}t_i)^{(p+1)^2}V^{k+1}(1,2,...,p+1)(\prod_{k+1\geq m>n\geq 1}(t_m-t_n))^{(p+1)^2}, T=t_{k+1},$$

where $V(1,2,...,p+1)$ is the Vandermonde Determinant for the numbers $1,2,...,p+1$. This means that the solution $c_{p+1},c_{p+2},...,c_{(p+1)(k+2)-1}$ of the system (7) exists and is unique. Thus there exists of coefficients vector $c_j,\ j=\overline{p+1,(p+1)(k+2)-1}$.

We have thus constructed vector - function $x_p(t)$ of the p-th step in the form of a polynomial according t of degree t $((p+1)(k+2)-1)$.

The 3-th equation of the system (3) when $i=p$ defines the function pseudocontrollibility $y_p(t)$ last step. Function $P_p y_p(t)$ is an element of the subspace $\ker D_p$, to be in view of the representation:

$$P_p y_p(t)=P_p(I_{p-1}-Q_{p-1})x_{p-1}(t)$$

to meet the $p(k+2)-$th conditions of the mind:

$$\begin{cases} \left.\frac{d^j P_p y_p(t)}{dt^j}\right|_{t=0}=P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j, \\ \left.\frac{d^j P_p y_p(t)}{dt^j}\right|_{t=t_1}=P_p(I_{p-1}-Q_{p-1})\gamma_{1,p-1}^j, \\ ... \\ \left.\frac{d^j P_p y_p(t)}{dt^j}\right|_{t=t_k}=P_p(I_{p-1}-Q_{p-1})\gamma_{k,p-1}^j, \\ \left.\frac{d^j P_p y_p(t)}{dt^j}\right|_{t=T}=P_p(I_{p-1}-Q_{p-1})\gamma_{k+1,p-1}^j, \\ j=\overline{1,p-1}. \end{cases} \tag{8}$$

This function is constructed in the form of a polynomial according t of degree $[p(k+2)-1]$ with coefficients vector:

$$P_p y_p(t)=\sum_{j=0}^{p(k+2)-1}l_j t^j. \tag{9}$$

Substituting (9) and its derivatives up to $(p-1)-$th order, in the relevant conditions of the system (8), we obtain the values of the coefficients $l_j,\ j=\overline{0,p-1}$:

$$l_j=\frac{1}{j!}P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j. \tag{10}$$

Expression (9) and its derivatives up to the $(p-1)-$th order, taking into account the other conditions (7) form a system of equations for the unknown $l_j,\ j=\overline{p,p(k+2)-1}$:

$$\begin{cases} t_1^p l_p+t_1^{p+1}l_{p+1}+...+t_1^{p(k+2)-1}l_{p(k+2)-1}= \\ =P_p(I_{p-1}-Q_{p-1})\gamma_{1,p-1}^0-\sum_{j=0}^{p-1}\frac{1}{j!}P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j t_1^j, \\ pt_1^{p-1}l_p+(p+1)t_1^p l_{p+1}+... \\ +[p(k+2)-1]t_1^{p(k+2)-2}l_{p(k+2)-1}= \\ =P_p(I_{p-1}-Q_{p-1})\gamma_{1,p-1}^1-\sum_{j=1}^{p-1}\frac{1}{(j-1)!}P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j t_1^j, \\ .... \end{cases}$$

$$\begin{cases} p(p-1)p...2t_1l_p+(p+1)p...3t_1^2l_{p+1}+... \\ +[p(k+2)-1][p(k+2)-2]...[p(k+1)+1]t_1^{p(k+1)}l_{p(k+2)-1} \\ =P_p(I_{p-1}-Q_{p-1})\gamma_{1,p-1}^{p-1}-P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^{p-1}, \\ t_2^p l_p+t_2^{p+1}l_{p+1}+...+t_2^{p(k+2)-1}l_{p(k+2)-1}= \\ =P_p(I_{p-1}-Q_{p-1})\gamma_{2,p-1}^0-\sum_{j=0}^{p-1}\frac{1}{j!}P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j t_2^j, \\ pt_2^{p-1}l_p+(p+1)t_2^p l_{p+1}+...+[p(k+2)-1]t_2^{p(k+2)-2}l_{p(k+2)-1}= \\ =P_p(I_{p-1}-Q_{p-1})\gamma_{2,p-1}^1-\sum_{j=1}^{p-1}\frac{1}{(j-1)!}P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j t_2^j, \\ .... \end{cases}$$

$$\begin{cases} t_k^p l_p + t_k^{p+1} l_{p+1} + \ldots + t_k^{p(k+2)-1} l_{p(k+2)-1} = \\ P_p(I_{p-1}-Q_{p-1})\gamma_{k,p-1}^0 - \sum_{j=0}^{p-1} \frac{1}{j!} P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j t_k^j, \\ p t_k^{p-1} l_p + (p+1) t_k^p l_{p+1} + \ldots + [p(k+2)-1] t_k^{p(k+2)-2} l_{p(k+2)-1} = \\ P_p(I_{p-1}-Q_{p-1})\gamma_{k,p-1}^1 - \sum_{j=1}^{p-1} \frac{1}{(j-1)!} P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j t_k^j, \\ \ldots. \\ p(p-1)p\ldots 2 t_k l_p + (p+1)p\ldots 3 t_k^2 l_{p+1} + \ldots \\ +[p(k+2)-1][p(k+2)-2]\ldots[p(k+1)+1] t_k^{p(k+1)} l_{p(k+2)-1} \\ = P_p(I_{p-1}-Q_{p-1})\gamma_{k,p-1}^{p-1} - P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^{p-1}, \end{cases}$$

$$\begin{cases} T^p l_p + T^{p+1} l_{p+1} + \ldots + T^{p(k+2)-1} l_{p(k+2)-1} = \\ P_p(I_{p-1}-Q_{p-1})\gamma_{k+1,p-1}^0 - \sum_{j=0}^{p-1} \frac{1}{j!} P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j T^j, \\ p T^{p-1} l_p + (p+1) T^p l_{p+1} + \ldots + [p(k+2)-1] T^{p(k+2)-2} l_{p(k+2)-1} = \\ P_p(I_{p-1}-Q_{p-1})\gamma_{k+1,p-1}^1 - \sum_{j=1}^{p-1} \frac{1}{(j-1)!} P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^j T^j, \\ \ldots. \\ p(p-1)p\ldots 2 T l_p + (p+1)p\ldots 3 T^2 l_{p+1} + \ldots \\ +[p(k+2)-1][p(k+2)-2]\ldots[p(k+1)+1] T^{p(k+1)} l_{p(k+2)-1} \\ = P_p(I_{p-1}-Q_{p-1})\gamma_{k+1,p-1}^{p-1} - P_p(I_{p-1}-Q_{p-1})\gamma_{0,p-1}^{p-1}. \end{cases} \quad (11)$$

The determinant Δ_2 of system (11) is set to:

$$\Delta_2 = \left(\prod_{i=1}^{k+1} t_i\right)^{p^2} V^{k+1}(1,2,\ldots,p)\left(\prod_{k+1 \ge m > n \ge 1} (t_m - t_n)\right)^{p^2}, T = t_{k+1},$$

where $V(1,2,\ldots,p)$ is the Vandermonde Determinant for the numbers $1,2,\ldots,p$. Thus the solution of the system (11) $l_p, l_{p+1}, \ldots, l_{p(k+2)-1}$ exists and is unique.

And so we construct a function $P_p y_p(t)$ in the form (9). Then, substituting in equation (5) a function $x_p(t)$ of (6), we obtain a function $y_p(t)$ of the $p-$th last step as a polynomial according t of degree $(p+1)(k+2)-1$ with coefficients vector.

Substituting the expression for $x_p(t)$ and $y_p(t)$ to the second equation of the system (3) when $i = p-1$, we obtain a function pseudo-states $x_p(t)$ in $(p-1)$-th step in the form of a polynomial according t of degree $p(k+2)-1$. And then from the 3-th equation of the system (3) when $i = p-1$, with regard to the expression for the function $x_{p-1}(t)$, we find the function pseudocontrollibility $y_{p-1}(t)$ penultimate $(p-1)$-th step in a polynomial according t with coefficients vector of degree $p(k+2)-1$. However, the function $P_{p-1}y_{p-1}(t)$ in the subspace $\ker D_p$ and satisfy $(p-1)(k+2)$ conditions:

$$\begin{cases} \left.\frac{d^j P_{p-1} y_{p-1}(t)}{dt^j}\right|_{t=0} = P_{p-1}(I_{p-2}-Q_{p-2})\gamma_{0,p-2}^j, \\ \left.\frac{d^j P_{p-1} y_{p-1}(t)}{dt^j}\right|_{t=t_1} = P_{p-1}(I_{p-2}-Q_{p-2})\gamma_{1,p-2}^j, \\ \ldots \\ \left.\frac{d^j P_{p-1} y_{p-1}(t)}{dt^j}\right|_{t=t_k} = P_{p-1}(I_{p-2}-Q_{p-2})\gamma_{k,p-2}^j, \\ \left.\frac{d^j P_{p-1} y_{p-1}(t)}{dt^j}\right|_{t=T} = P_{p-1}(I_{p-2}-Q_{p-2})\gamma_{k+1,p-1}^j, \\ j = \overline{1, p-2} \end{cases}$$

and in a polynomial according t of degree $(p-2)(k+2)-1$.

Furthermore, acting by induction, from the 2-th equation of system (3) we find the function pseudo-states $x_i(t)$, i-th step of decomposition in the form of polynomial according t of degree $(p+1)(k+2)-1$ t with coefficients vector, and from the 3-th equation of the system, taking into account the expressions for the $x_i(t)$, corresponding functions pseudocontrollibility $y_i(t)$ i-th step also in the form of polynomials according t of degree $(p+1)(k+2)-1$ with coefficients vector. At each i-th step function $y_i(t)$ will contain an element $P_i y_i(t)$ in subspace $\ker D_i$ and satisfy $(i+1)(k+2)$ conditions:

$$\begin{cases} \left.\frac{d^j P_i y_i(t)}{dt^j}\right|_{t=0} = P_i(I_{i-1}-Q_{i-1})\gamma_{0,i-1}^j, \\ \left.\frac{d^j P_i y_i(t)}{dt^j}\right|_{t=t_1} = P_i(I_{i-1}-Q_{i-1})\gamma_{1,i-1}^j, \\ \ldots \\ \left.\frac{d^j P_i y_i(t)}{dt^j}\right|_{t=t_k} = P_i(I_{i-1}-Q_{i-1})\gamma_{k,i-1}^j, \\ \left.\frac{d^j P_i y_i(t)}{dt^j}\right|_{t=T} = P_i(I_{i-1}-Q_{i-1})\gamma_{k+1,i-1}^j, \\ j = \overline{0, i}, \end{cases}$$

which is a polynomial according t with coefficients vector $i(k+2)-1$.

Thus, in the last step, taking into account the expressions for the functions $x_1(t)$ and $y_1(t)$, from the second equation (3) when $i=1$, we find the function $x(t)$ of the source system (1) as a polynomial according $t$with coefficients vector $(p+1)(k+2)-1$.

The 3-th equation of (3) when $i=1$, after substituting the expressions for $x(t)$, giving expression to controllibility function $u(t)$ of the original system (1). Function $Pu(t)$, as a member of the term in the formula for $u(t)$, the subspace $\ker D$ is chosen at random and it imposed no restrictions.

The theorem is proved.

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