# NON-ABELIAN SOLUTIONS OF THE YANG-MILLS EQUATIONS FOR THE SU(2) GAUGE FIELD COUPLED WITH THE TWO HIGGS FIELDS

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**Abstract -** This article considers the SU(2) gauge field coupled with the two Higgs fields. We obtain the non- Abelian solutions of the corresponding field equations. For the constant solution, the gauge field and the two Higgs fields correspond to unbroken local SU(2) symmetry at large distance. Our exact classical solution is infinite at $r = 1/\alpha$ and exhibits a form of an SU(2) gauge charge confinement. In the case constant $\alpha$ in our exact classical solution equals zero, the spatial component $\left(A_i^a\right)$ of the SU(2) gauge field is analogous to the potential of a point magnetic monopole. The constant $\beta$ in our exact solution is a mixing angle which determines the relative contribution of the two Higgs fields. We then investigate the energy of the gauge field and the two Higgs fields.

**Key words -** SU(2) gauge field, non-Abelian solution, two Higgs fields, Yang-Mills theory, SU(2) group, Yang-Mills equation.

## 1. Introduction

The non-Abelian gauge theory was invented by Yang and Mills [1]. This theory offer the greatest prospect to describe the elementary forces in nature. The Weinberg-Salam model (or generalizations of it) and quantum chromodynamics are the two quantum Yang-Mills theories of real phenomenogical importance [2, 3]. These theories can be formulated in terms of Feynman path integrals, i.e., functional integrals over all classical field configurations weighed by a factor exp(-action). If one knows everything about classical field configurations, then in principle all questions concerning the quantum theory could be answered. Partial information about classical fields might yield, at least, some insight into the quantum theory. This is the basic hope which motivates present research activity in classical Yang-Mills theory.

The classical Yang-Mills equations have interesting solution classes: monopole and dyon solutions, instanton and meron solutions, vortex solutions,… [4-8]. The solutions of the classical Yang-Mills theory have important role in quantum field theory. For example, instantons are solutions in Euclidean space-time, and this means they have something to do with tunneling in Minkowski spase in the quantum Yang-Mills theory. Imaginary-time solutions of classical theories are usually interpreted as real-time tunneling in the corresponding quantized theory. One important result that instantons seem to be connected with quark confinement. The unknown mechanism which keeps the quarks and gluons inside hadrons.

This paper presents some non-Abelian solutions of Yang-Mills equations for the SU(2) gauge field coupled with the two Higgs fields. In Section 2, we give Yang-Mills equations of this fields. The classical solutions of corresponding field equations are considered in Section 3. One of these solutions exhibits the property of confinement that has been looked for in non-Abelian gauge theories. Section 4 investigates the energy of the gauge field and the two Higgs fields. The discussions and conclusions are given in Section 5.

## 2. Yang-Mills Equations of the SU(2) Gauge Field Coupled with the Two Higgs Triplets

We consider the model of the SU(2) gauge field coupled with the two Higgs fields. The Lagrangian density for this system is

$$\begin{aligned} \mathscr{L} = & -\frac{1}{4}F_{\mu\nu}^{a}F^{\mu\nu a} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a + \frac{1}{2}D_\mu\psi^a D^\mu\psi^a \\ & -\frac{\lambda}{4}\left(\frac{m^2}{\lambda}-\phi^2\right)^2 - \frac{\lambda'}{4}\left(\frac{m'^2}{\lambda'}-\psi^2\right)^2 \\ & -\frac{1}{2}g^2\left[\phi^2\psi^2-\left(\phi^a\psi^a\right)^2\right], \end{aligned} \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon^{abc}A_\mu^b A_\nu^c, \quad (2)$$

$$D_\mu\phi^a = \partial_\mu\phi^a + g\varepsilon^{abc}A_\mu^b\phi^c, \quad (3)$$

$$D_\mu\psi^a = \partial_\mu\psi^a + g\varepsilon^{abc}A_\mu^b\psi^c, \quad (4)$$

$A_\mu^a$ and $F_{\mu\nu}^a$ are the SU(2) non-Abelian gauge potential and field intensity tensor, respectively; $\phi^a$ and $\psi^a$ are the two Higgs fields; $\mu, \nu = 0, 1, 2, 3$ are space-time indices; $a, b, c = 1, 2, 3$ are SU(2) group indices; $m, \lambda, m', \lambda'$ are parameters; $g$ is the coupling constant. Notice that the coupling of $\phi^a$ and $\psi^a$ is given by the last term in the Lagrangian density (1). For this term to vanish, $\phi^a$ and $\psi^a$ have to have the same direction in group space.

We are interested in static solution, so all the time derivatives will be zero. For the case two Higgs fields have the same direction, from the Lagrangian density (1) when using the Lagrange-Euler equation we obtain the motion equations for the SU(2) gauge and the two Higgs fields. The gauge field equations have the form

$$\begin{aligned} &\partial_i F^{\mu i a} + g\varepsilon^{abc}A_i^b F^{\mu i c} \\ &\quad = g\varepsilon^{abc}\left(D^\mu\phi^b\right)\phi^c + g\varepsilon^{abc}\left(D^\mu\psi^b\right)\psi^c. \end{aligned} \quad (5)$$

For two Higgs fields the equations are

$$\partial_i\left(D^i\phi^a\right) + g\varepsilon^{abc}A_\mu^b\left(D^\mu\phi^c\right) = -m^2\phi^a + \lambda\phi^2\phi^a, \quad (6)$$

$$\partial_i\left(D^i\psi^a\right) + g\varepsilon^{abc}A_\mu^b\left(D^\mu\psi^c\right) = -m'\psi^a + \lambda'\psi^2\psi^a. \quad (7)$$

Assuming that the gauge field and the two Higgs fields are radial, we use the Wu-Yang ansatz [9]

$$A_i^a = \varepsilon_{aij}\frac{r^j}{gr^2}\left[1-H(r)\right], \quad A_0^a = 0,$$

$$\phi^a = \frac{r^a}{gr^2}I(r), \quad \psi^a = \frac{r^a}{gr^2}J(r). \quad (8)$$

Inserting this ansatz into the equations (6) and (7) yields three coupled nonlinear differential equations

$$r^2H'' = H\left(H^2+I^2+J^2-1\right),$$
$$r^2I'' = 2IH^2 + \frac{\lambda}{g^2}\left(I^3 - \frac{m^2g^2}{\lambda}r^2I\right), \quad (9)$$
$$r^2J'' = 2JH^2 + \frac{\lambda'}{g^2}\left(J^3 - \frac{m'^2g^2}{\lambda'}r^2J\right),$$

where $H''$, $I''$ and $J''$ denote differentiation with respect to *r*.

## 3. Solutions of the Field Equations

We discuss some properties of the coupled differential equations (9). For $m^2 \neq 0$, $m'^2 \neq 0$, $\lambda \neq 0$, $\lambda' \neq 0$ they evidently cannot be solved analytically, but it can be obtained numerically. For $m = 0$, $m' = 0$ and arbitrary $\lambda$, $\lambda'$ there is a constant solution of this equation system, namely

$$H = -\frac{\lambda'}{2g^2\left(1+\frac{\lambda'}{\lambda}-\frac{\lambda'}{2g^2}\right)},$$
$$I = -\frac{\lambda'}{\lambda\left(1+\frac{\lambda'}{\lambda}-\frac{\lambda'}{2g^2}\right)}, \quad (10)$$
$$J = -\frac{1}{1+\frac{\lambda'}{\lambda}-\frac{\lambda'}{2g^2}}.$$

The gauge and Higgs fields in this case correspond to unbroken local SU(2) symmetry ($\phi^a \to 0, \psi^a \to 0$ at infinity).

When $m^2 \to 0$, $\lambda \to 0, m'^2 \to 0, \lambda'^2 \to 0$ and $m^2/\lambda$ and $m'^2/\lambda'$ finite, the equation system (9) is reduced to

$$r^2H'' = H\left(H^2+I^2+J^2-1\right),$$
$$r^2I'' = 2IH^2, \quad (11)$$
$$r^2J'' = 2JH^2.$$

The exact solution of the motion equations of the SU(2) gauge field coupled with one massless Higgs field was found by Singleton [10]. Here we consider the SU(2) gauge field coupled with the two Higgs fields. The exact classical solution of the equation system (11) is

$$H(r) = \frac{\alpha r}{\alpha r - 1},$$
$$I(r) = \pm\frac{\sin\beta}{\alpha r - 1}, \quad (12)$$
$$J(r) = \pm\frac{\cos\beta}{\alpha r - 1},$$

where $\alpha$ and $\beta$ are arbitrary constants. Inserting $H(r)$, $I(r)$, and $J(r)$ into the expressions for the gauge field and the two Higgs fields of ansatz (8), we obtain

$$A_i^a = \pm\varepsilon_{aij}\frac{r^j}{gr^2}\left(1-\frac{\alpha r}{\alpha r-1}\right),\ A_0^a = 0,$$
$$\phi^a = \pm\frac{r^a\sin\beta}{gr^2(\alpha r-1)},\ \psi^a = \pm\frac{r^a\cos\beta}{gr^2(\alpha r-1)}. \quad (13)$$

It is seen that the gauge field and the two Higgs fields become infinite at the radius

$$r = r_0 = \frac{1}{\alpha}. \quad (14)$$

From (13) we see that if $\beta = 0$ then $\phi^a = 0$, $\psi^a = \pm\frac{r^a}{gr^2(\alpha r-1)}$; if $\beta = \pi/2$ then $\phi^a = \pm\frac{r^a}{gr^2(\alpha r-1)}$, $\psi^a = 0$. Thus, $\beta$ is a mixing angle which determines the relative contributions of the two Higgs fields. In the case $\alpha = 0$, the spatial component $\left(A_i^a\right)$ of the gauge field in the solution (13) becomes

$$A_i^a = \pm\varepsilon_{aij}\frac{r^j}{gr^2}. \quad (15)$$

The solution (15) is like to the potential of a point magnetic monopole.

The exact classical solution (13) is similar for the Schwarzschild solution in general relativity [11]. Using these gauge potentials to calculate the non-Abelian electromagnetic field intensities, we find:

$$E_i^a = F_{0i}^a = 0,$$
$$B_i^a = -\frac{1}{2}\varepsilon_{ijk}F_{jk}^a = \frac{1}{g}\left[\frac{(1-2\alpha r)\hat{r}^i\hat{r}^a}{r^2(\alpha r-1)^2} + \frac{\alpha\left(\delta^{ia}-\hat{r}^i\hat{r}^a\right)}{r(\alpha r-1)^2}\right], \quad (16)$$

where $\hat{r}^i$ and $\hat{r}^a$ are the unit vectors. The non-Abelian magnetic field intensity is also infinite at $r = r_0 = 1/\alpha$.

We see that the SU(2) non-Abelian gauge field and the two Higgs fields also become singular at $r = 0$, which is true as well for the Coulomb potential of a point charge in classical electromagnetism. The singularities of all these solutions at $r = 0$ are of the same character in that they all imply a $\delta$ function point charge sitting at the origin (where the charge of our non-Abelian model is an SU(2) gauge charge). The singularity in our solution at $r = r_0$ has a different character than the singularity in the Schwarzschild solution. The singularity in the Schwarzschild solution is not a true singularity, which arises because of the choice of coordinates. For our classical solution, the singularity at $r = r_0$ is a real singularity.

## 4. Energy Expression

The energy of the SU(2) non-Abelian gauge field coupled with the two Higgs fields can be obtained by taking the volume integral of the time-time component of the energy-momentum tensor

$$E = \int T^{00}d^3x. \quad (17)$$

where $T^{00}$ is energy density, which is defined by the energy-momentum tensor

$$T^{\mu\nu} = F^{\mu\rho a}F_\rho^{\nu a} + D^\mu\phi^a D^\nu\phi^a + D^\mu\psi^a D^\nu\psi^a + g^{\mu\nu}\mathcal{L} \quad (18)$$

When two Higgs fields have the same direction and in the limit $m^2 \to 0$, $\lambda \to 0$, $m'^2 \to 0$, $\lambda'^2 \to 0$, $m^2/\lambda$ and $m'^2/\lambda'$ finite, we obtain

$$E = \frac{4\pi}{g^2}\int_{r_\alpha}^{\infty}\left[H'^2 + \frac{\left(H^2-1\right)^2}{2r^2} + \frac{I^2H^2}{r^2} + \frac{\left(rI'-I\right)^2}{2r^2} + \frac{J^2H^2}{r^2} + \frac{\left(rJ'-J\right)^2}{2r^2}\right]dr. \quad (19)$$

Notice that the integral has been cut off from below at an arbitrary distance $r_\alpha$, which must be large than $r = r_0$. This procedure is done to avoid the singularities at $r = 0$ and $r = r_0$, since integrating through $r = 0$, $r_0$ would give an infinite field energy. This is similar to the Coulomb potential of point electric charge, which yields an infinite field energy when integrated down to zero. An additional argument for introducing the cut off $r_\alpha$ is the fact that our classical solution does not exhibit asymptotic freedom. In this respect $r_\alpha$ could be taken to delineate the boundary between the region where our classical solution dominates and the region where the quantum effect of asymptotic freedom dominates. Inserting $H(r)$, $I(r)$ and $J(r)$ in the solution (12) into the equation (19) we find

$$E = \frac{2\pi}{g^2}\left(\sin^2\beta + \cos^2\beta + 1\right)\frac{\left(2\alpha r_\alpha - 1\right)}{r_a\left(\alpha r_\alpha - 1\right)} = \frac{4\pi\left(2\alpha r_\alpha - 1\right)}{g^2 r_\alpha\left(\alpha r_\alpha - 1\right)}. \quad (20)$$

The energy of our classical solution is finite.

## 5. Discussions and Conclusions

In the Lagrangian density (1), the Higgs fields must be nonvanishing at spatial infinity in order that the potential energy be zero there. Thus any physical solution must satisfy

$$\phi_a \to \frac{m}{\sqrt{\lambda}}\hat{n}_a,\ \psi_a \to \frac{m'}{\sqrt{\lambda'}}\hat{n}_a,\quad r \to \infty, \quad (21)$$

where $\hat{n}_a$ is unit vector. By assumption, there is vacuum at spatial infinity in this classical case. This is like spontaneous symmetry breaking in the quantum theory, where one gives the Higgs fields a nonzero vacuum expectation value: $<\phi>\neq 0, <\psi>\neq 0$.

The singularity of the SU(2) non-Abelian magnetic field intensity (15) at $r = r_0$ seems to exhibit a form of SU(2) gauge charge confinement. An SU(2) gauge charge carring particle, which enter the region $r < r_0$, is not able to leave this region. This is analogous to what happens with the Schwarzschild solution in general relativity, where once a particle passes the event horizon it is permanently confined. We see that, as $r \to \infty$, the magnetic field inrensity fall off like $\left(1/r^3\right)$, unlike the Prasad-Sommerfield solution, which has a $\left(1/r^2\right)$ behavior for lagre $r$ [12].

Just as the singularity at the origin can be taken to be a point source of the SU(2) gauge charges, the singularity at $r = r_0$ can be taken to be a spherical shell of the SU(2) gauge charges. Therefore $r_0$ should be roughly related to the radius of the SU(2) gauge charge bound states. Using the analogy between Yang-Mills theory and general relativity it can be argued that constant $\alpha$ should be related to the streng of the interaction and the magnitude of the SU(2) gauge charges.

Our classical solution is for an SU(2) gauge theory, while quantum chromodynamics is formulated in terms of the SU(3) gauge group. Thus, one would need to generalize the present solution to SU(3) or, if possible, to SU(N). This should be possible by embedding the SU(2) solution in the higher rank gauge group.

In summary, by direct calculation we have given two explicit solutions of the Yang-Mills equations for the gauge field coupled with the two Higgs fields. For the constant solution, the gauge and Higgs fields correspond to unbroken local SU(2) symmetry at large distance. Our exact classical solution exhibits the property of the SU(2) gauge charge confinement. If in the exact classical solution constant $\alpha = 0$, then the spatial component $\left(A_i^a\right)$ of the SU(2) gauge field has a form of the potential of point magnrtic monopole. The energy of the gauge field and the two Higgs fields is finite.

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