FREE VIBRATION OF FUNCTIONALLY GRADED BEAMS RESTING ON WINKLER FOUNDATION AROUND THE BUCKLING DOMAIN

DAO ĐỘNG TỰ DO CỦA DẦM CÓ ĐẶC TÍNH BIẾN THIÊN TRÊN NỀN WINKLER XUNG QUANH MIỀN MẤT ỔN ĐỊNH

Huynh Vinh, Do Minh Duc

University of Science and Technology - The University of Danang; ducdhbk@gmail.com

Abstract - This paper investigates free vibration of functionally graded beams (FG beams) resting on Winkler foundation around the buckling domain. The governing equations of motion based on Euler–Bernoulli and Timoshenko beam theories together with Von Karman’s strain–displacement relation are derived from Lagrange’s equations. The material properties of the beams vary continuously in the thickness direction according to the power-law distributions. The finite element method is employed to discretize the model and obtain numerical approximations of the equilibrium equations. Numerical examples are carried out with the Matlab programming language. The reliability of the solution is verified by comparison with previously reported results in the literature. The effect of shear deformation, boundary conditions, and different material property distributions on the first frequency of the beams are also analyzed and discussed.

Key words - free vibration; FG beams; finite element method; critical buckling loads; Winkler foundation.

Tóm tắt - Bài báo này phân tích dao động riêng của dầm có đặc tính biến thiên (dầm FG) trên nền Winkler quanh miền mất ổn định. Hệ phương trình chuyển động của dầm dựa trên lý thuyết dầm Euler–Bernoulli và Timoshenko với quan hệ giữa biến dạng và chuyển vị Von Karman được thiết lập theo phương trình Lagrange. Đặc tính của vật liệu giả thiết thay đổi liên tục theo chiều cao dầm theo quy luật hàm lũy thừa. Phương pháp phần tử hữu hạn được sự dụng để rời rạc mô hình tính toán và tìm nghiệm số xấp xỉ của hệ phương trình. Các ví dụ số được thực hiện với sự trợ giúp của phần mềm Matlab. Độ tin cậy của lý thuyết phát triển được kiểm tra bằng cách so sánh với các kết quả đã công bố. Ảnh hưởng của biến dạng cắt, các điều kiện liên kết ở biên cũng như sự phân bố của đặc tính vật liệu khác nhau đến tần số dao động đầu tiên của dầm cũng được phân tích và bàn luận.

Từ khóa - dao động tự do; dầm FG; phương pháp phần tử hữu hạn (PTHH); lực tới hạn; nền Winkler.

1. Introduction

Composite materials are often combined with two or more materials having significantly different properties to exploit good characteristics from the individual components. Compared to traditional materials, these materials are stronger, lighter, or less expensive; thus, they have widely been used for buildings, bridges, structures, element machines, etc. However, the sudden change in material properties at the interface between the layers causes stress concentration and separation of the layers [9]. One of the solutions to this undesired problem is the use of Functional Graded Materials (FGMs).

Functional Graded Materials were first successfully made in the mid-1980s by Japanese scientists [9]. Unlike laminated composite, they are blended from constituent materials in accordance with the desired mathematical model continuously. This leads to a gradual change of the mechanical properties of FGMs; thus, reducing the thermal stresses, eliminating concentrate stresses and residual stresses. FGMs also have outstanding advantages: high strength, low weight, ability to withstand high-temperatures, good corrosion resistance of the environments, etc. FGMs are used in very different applications, such as reactor vessels, fusion energy devices, biomedical sectors, aircrafts, space vehicles, defense industries and other engineering structures [12].

In dynamic response analysis of structures, determining natural vibration and identifying its characteristics are very important work. This problem always receives a lot of attention from scientists. There have been numerous reports on the linear vibration of FG beams published in the scientific literature and can be listed here a number of most recent studies, such as [3-4, 8, 10-12]. Many reports also took into consideration the nonlinear geometric effect in analysis in order to reflect more accurately the dynamic response [5-6, 14-15, 17-18]; the published results showed that the geometric nonlinearity had significant effects, and in many cases there were even large differences when compared to the results obtained from linear analysis. This is due to the change in stiffness of structures caused by this effect. Obviously, the problem becomes more complicated if the stiffness of a beam approaches zero, which means that the beam is in buckling regions. To the best knowledge of the authors, there are very few reports [7, 13] on dynamic response of FG beams in pre- and post-buckling domain, and no report of these on elastic environment is found.

The aim of this paper is to investigate free vibration of FG beams resting on Winkler foundation around the buckling domain. The governing equations of motion based on Euler-Bernoulli and Timoshenko beam theories together with Von Karman’s strain-displacement relation are derived from Lagrange’s equations. The material properties of the beams vary continuously in the thickness direction according to the power-law distributions. The finite element method is exploited to discretize the model and obtain numerical approximations of the equilibrium equations. Numerical examples are carried out by the Matlab programming language. The reliability of the solution is verified by comparison with previously reported results in the literature. The influences of shear deformation, boundary conditions, and different material property distributions on the first frequency of the beams are also analyzed and discussed.

2. Basic mathematical equations

Consider a straight uniform functionally graded beam of length L, width b, thickness h, with coordinate system

(O*xyz*) having the origin O, see Figure 1, where the *x*-axis coincides with the mid-plane; the *y*-axis in the width direction and the *z*-axis in the depth direction. The beam is subjected to axial centric forces *P* at the two ends and placed on Winkler foundation with spring constants k_w.

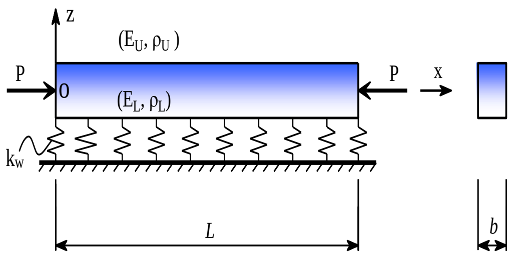

Figure 1. *An FG beam resting on Winkler elastic foundation*

2.1. Material properties

In this study, it is assumed that the effective material properties of the beam, i.e., Young's modulus (*E*), Poisson's ratio (ν), shear modulus (*G*) and mass density (ρ) - commonly denoted *T*, vary continuously in the thickness direction (*z*-axis) according to power-law form.

According to the rule of mixture [9], the effective material properties (*T*) can be expressed as:

$$T(z) = T_L V_L + T_U V_U \quad (1)$$

where T_L, T_U and V_L, V_U are the corresponding material properties and the volume fractions of two materials that compose the beam.

V_U and V_L are related by following equation:

$$V_U + V_L = 1 \quad (2)$$

Assuming that the V_U is varied by power law distribution as follows [9]:

$$V_U = \left(\frac{z}{h} + \frac{1}{2}\right)^k \quad (3)$$

where *k* is the non-negative variable parameter (power-law exponent) which dictates the material variation profile through the thickness of the beam.

Therefore, substituting Eq. (3) into Eq. (2) then substituting all into Eq. (1), the effective material properties (*T*) can be described as follows:

$$T(z) = (T_U - T_L)\left(\frac{z}{h} + \frac{1}{2}\right)^k + T_L \quad (4)$$

It can be seen from Eq. (4) that

- at $z = -h/2$, $T = T_L$ and at $z = h/2$, $T = T_U$
- $k = 0$, $T = T_U$ and the beam becomes homogeneous beam consisting only *U* component.

2.2. The displacements and deformations on the beam:

In this study, both Euler-Bernoulli and Timoshenko beam theories are used to consider and evaluate the effect of shear deformation on the behaviors of the beam.

Based on Timoshenko beam theory (FOBT), the axial displacement, *u*, and the transverse displacement of any point of the beam, *w*, can be expressed in matrix form as below:

$$\{d\} = \begin{Bmatrix} u(x,z,t) \\ w(x,z,t) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & \left(-z + \Phi(z)\right) & -\Phi(z) \\ 0 & 1 & 0 & 0 \end{bmatrix} \{A_1\} \quad (5)$$

$$\{A_1\}^T = \left\{ u_o(x,t) \quad w_o(x,t) \quad \frac{\partial w_o(x,t)}{\partial x} \quad \phi_o(x,t) \right\} \quad (6)$$

where u_o and w_o are the axial and the transverse displacement of any point on the mid-plane corresponding to displacement *u* and *w*, respectively; *t* denotes time; ϕ_o is the total bending rotation of the cross sections at any point on the neutral axis; $\Phi(z) = z$ for Timoshenko beam theory and $\Phi(z) = 0$ for Euler-Bernoulli beam theory (EBT) [12].

The normal strain in the *x*-direction, ε_x, through the Von-Karman type nonlinear strain–displacement relations, and the shear strain, γ_{xz}, -displacement relations of the beam at a distance *z* can be represented by Eq. (7):

$$\begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & \left(-z + \Phi(z)\right) & -\Phi(z) & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\partial \Phi(z)}{\partial z} & -\dfrac{\partial \Phi(z)}{\partial z} \end{bmatrix} \{A_2\}}_{\{\varepsilon_1\}} + \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \{A_3\}}_{\{\varepsilon_2\}} \quad (7)$$

where

$$\{A_2\}^T = \left\{ \frac{\partial u_o(x,t)}{\partial x} \quad \frac{\partial^2 w_o(x,t)}{\partial x^2} \quad \frac{\partial \Phi_o(x,t)}{\partial x} \quad \frac{\partial w_o(x,t)}{\partial x} \quad \phi_o(x,t) \right\} \quad (8)$$

$$\{A_3\}^T = \left\{ \left(\frac{\partial u_o(x,t)}{\partial x}\right)^2 \quad \left(\frac{\partial w_o(x,t)}{\partial x}\right)^2 \right\} \quad (9)$$

$\{\varepsilon_1\}$ and $\{\varepsilon_2\}$ are, respectively, the linear and the nonlinear strain.

2.3. The strain-stress relation

Assuming that the material of the FG beam obeys Hooke's law, the strain–stress relationship equation can be written as Eq. (10) below [2]:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} E(z) & 0 \\ 0 & k_s G(z) \end{bmatrix} . \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} = [E] . \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} \quad (10)$$

where σ_x is the axial normal stress, τ_{xz} is the shear stress, and k_s is the shear correction factor, which is taken as 5/6 for rectangular cross section.

2.4. The governing equations of motion

The equations of motion are derived from Lagrange's equations as follows [1]:

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}}\right\} - \left\{\frac{\partial L}{\partial q}\right\} + \left\{\frac{\partial R}{\partial \dot{q}}\right\} = \{0\} \quad (11)$$

$$L = W - \pi \quad (12)$$

where *L* is the Lagrange function, *W* is the kinetic energy, π is the potential energy, *R* is the dissipation function which is often neglected in free vibration analysis, *q* is the nodal displacement, and $\dot{q}$ is the nodal velocity.

In this study, finite element method (FEM) is employed to obtain the approximate displacement field. The beam is discretized into elements. For each element, the potential and the kinetic energy can be expressed as:

$$\pi_e = \frac{1}{2}\int_{V_e} \{\varepsilon_1\}_e^T \{\sigma\}_e \, dV + \frac{1}{2}\int_0^l k_w.w_o^T.w_o.dx + P\int_o^l \{\varepsilon_2\}.dx \quad (13)$$

Applying Hook' law to the linear strain ε_1 leads to:

$$\pi_e = \frac{1}{2}\int_{V_e} \{\varepsilon_1\}_e^T [E]\{\varepsilon_1\}_e \, dV + \frac{1}{2}\int_0^l k_w.w_o^T.w_o.dx + P\int_o^l \{\varepsilon_2\}.dx \quad (14)$$

By substituting Eqs. (7-9) into Eq. (14), we obtain Eq. (15)

$$\pi_e = \frac{1}{2}\int_0^l \{A_2\}^T.[D]_E.\{A_2\}dx + \frac{1}{2}\int_0^l k_w.w_o^T.w_o.dx + \frac{1}{2}P\int_o^l \left(\frac{\partial u_o^T}{\partial x}.\frac{\partial u_o}{\partial x} + \frac{\partial w_o^T}{\partial x}.\frac{\partial w_o}{\partial x}\right)dx \quad (15)$$

where $[D]_E$ is the square matrix of the elastic coefficients ($E(z)$, $G(z)$)

$$W_e = \frac{1}{2}\int_{V_e} \rho.\{\dot{d}\}_e^T.\{\dot{d}\}_e.dV \quad (16)$$

By substituting Eq. (5) into Eq. (16) we obtain:

$$W_e = \frac{1}{2}\int_0^l \{\dot{A}_1\}^T.[D]_R.\{\dot{A}_1\}dx \quad (17)$$

$$\{\dot{A}_1\}^T = \left\{\dot{u}_o(x,t) \quad \dot{w}_o(x,t) \quad \frac{\partial \dot{w}_o(x,t)}{\partial x} \quad \dot{\phi}_o(x,t)\right\}$$

where $[D]_R$ is the matrix containing the mass density (ρ)

In the equations described above, l and V_e is the length and the volume of the beam element respectively; "e" denotes the eth-element.

Approximate displacement of any point inside the element to the nodal displacements at the nodes through the shape functions N_i [1]:

$$u_o^e = \Sigma N_i(x).q_e\,;\; w_o^e = \Sigma N_i(x).q_e\,;\; \phi_o^e = \Sigma N_i(x).q_e \quad (18)$$

where $\{q\}_e^T = \{u_i, w_i, \varphi_i, \phi_i, u_j, w_j, \varphi_j, \phi_j\}$: the element nodal displacement vector (four degrees of freedom per node model is used)

By substituting Eq. (18) into Eq. (15) and (17), the potential and the kinetic energy are expressed through the nodal displacements, then substituting them in to Eq. (11) leads to the equations of motion of the element for free vibration which is described in the matrix form as below:

$$[M]_e\{\ddot{q}\}_e + ([K]_e + [K_G]_e + [K_F]_e)\{q\}_e = \{0\} \quad (19)$$

where $[K]_e, [K_G]_e, [K_F]_e$ are, respectively, the element elastic stiffness matrix, the element geometric stiffness matrix, the element stiffness matrix due to the elastic foundation, and $[M]_e$ the element mass matrix.

By assembling all the element equations, the global equations of motion of the beam can be written as:

$$[M]\{\ddot{q}\} + ([K] + [K_G] + [K_F])\{q\} = \{0\} \quad (20)$$

where $[K], [K_G], [K_F]$ are the global stiffness matrices of the beam.

The differential Eq. (20) can be solved by assuming the following type of solution:

$$\{q\} = \{\bar{q}\}e^{i\omega t} \quad (21)$$

where ω is the natural frequency of the beam.

Then, the equations become the following eigenvalue problem:

$$([K] + [K_G] + [K_F] - \omega^2[M])\{\bar{q}\} = \{0\} \quad (22)$$

Note that the critical buckling loads (P_{cr}) are also found by setting the mass matrix in Eq. (22) to zero (M = 0).

3. Numerical results and discussions

In this section, free vibration responses of the axially loaded FG beam placed on Winkler foundation as shown in Figure 1 are investigated. Based on the theory presented above, a computational program is written in the Matlab programming language to solve the governing equations and obtain the numerical results. Different boundary conditions, such as clamped-clamped (C-C), clamped-free (C-F), simply supported (S-S), clamped-hinged (C-H) end conditions, are considered. The width of the beam is taken to be unity.

The FG beam is assumed to be composed of aluminum (Al) and alumina (Al_2O_3) - referred from Ref. [12]. Herein, the bottom surface of the beam is pure aluminum (E_L = 70 GPa, ρ_L = 2702 kg/m^3, ν_L = 0.3) whereas the top surface of the beam is pure alumina (E_U = 380 GPa, ρ_U = 3960 kg/m^3, ν_U = 0.3)

In order to facilitate the presentation of results, dimensionless parameters are used as Eqs. (23) below:

$$\lambda = \frac{\omega L^2}{h}\sqrt{\frac{\rho_L}{E_L}}\,;(\bar{P}_{cr}, \bar{P}) = (P_{cr}, P).\frac{12L^2}{E_L h^3}\,;\eta = \frac{P}{P_{cr}}\,;K_w = \frac{k_w.L^4}{E_L h^3} \quad (23)$$

3.1. Model verification

With the aim to verify the accuracy of the present study, this subsection analyzes responses of the beam without Winkler foundation having the parameters as described above which was investigated by Simsek [12] for free linear vibration analysis and by Li and Batra [16] for buckling analysis. The results obtained in this study agree well with those reported in Refs. [12] and [16] as shown in Tables 1 and 2. It should be noted that Poisson's ratios (ν_U, ν_L) were taken to be 0.23 and the nonlinear deformation caused by the axial displacement u_o was not considered in Ref. [16] for the buckling analysis; thus, the solution proposed in this study gives more accurate results.

3.2. Results and discussions

This subsection is devoted to presenting new results and discussing the effect of the parameters on response of the beam.

The results in Table 3 and Figs. 2-5 show that as the value of the applied load P increases, the natural frequency decreases in the pre-buckling domain whereas this trend becomes reverse in the post-buckling domain; the natural frequency is zero when P is equal to the critical buckling load. It is because the variation of P leads to the change in the stiffness of the beam. Note that the closer P is to the position critical buckling load, the faster the speed of variation of the frequency value is, and the results are almost symmetric with respect to this position. It is also noteworthy that the buckling load could be determined from the condition of zero frequency.

Table 1. *Comparison of the non-dimensional critical buckling loads $\bar{P}_{cr}$ of FG beams (L/h = 5).*

Theory	Boundary condition	Reference	k					
			0	0.5	1	2	5	10
FOBT	C-C	Li and Batra [16]	154.3500	103.2200	80.4980	62.6140	50.3840	44.2670
		Present(*)	154.3602	103.2296	80.5025	62.6183	50.3869	44.2695
		Present	154.3602	102.6817	79.4743	61.2868	49.3668	43.7426
	S-S	Li and Batra [16]	48.8350	31.9670	24.6870	19.2450	16.0240	14.4270
		Present(*)	48.8361	31.9678	24.6876	19.2454	16.0243	14.4274
		Present	48.8361	31.9066	24.5754	19.0987	15.9011	14.3595
	C-F	Li and Batra [16]	13.2130	8.5782	6.6002	5.1495	4.3445	3.9501
		Present(*)	13.0770	8.4992	6.5427	5.1040	4.2985	3.9031
		Present	13.0770	8.4947	6.5344	5.0932	4.2891	3.8978

(*) *This item indicates the solution without consideration of the nonlinear deformation caused by the u_o component*

Table 2. *Comparison of the non-dimensional fundamental natural frequencies λ of FG beams (L/h = 5).*

Theory	Boundary condition	Reference	k					
			0	0.5	1	2	5	10
FOBT	C-C	Simsek [12]	10.0344	8.7005	7.9253	7.2113	6.6676	6.3406
		Present	10.0014	8.6734	7.9020	7.1895	6.6443	6.3167
	S-S	Simsek [12]	5.1525	4.4083	3.9902	3.6344	3.4312	3.3134
		Present	5.1525	4.3989	3.9709	3.6047	3.4023	3.2961
	C-F	Simsek [12]	1.8948	1.6174	1.4630	1.3338	1.2645	1.2240
		Present	1.8945	1.6170	1.4628	1.3336	1.2642	1.2237
EBT	C-C	Simsek [12]	12.1826	10.3718	9.3642	8.5277	8.1096	7.8797
		Present	12.1826	10.3697	9.3619	8.5235	8.1020	7.8723
	S-S	Simsek [12]	5.3953	4.5936	4.1484	3.7793	3.5949	3.4921
		Present	5.3953	4.5819	4.1246	3.7429	3.5579	3.4691
	C-F	Simsek [12]	1.9385	1.6506	1.4914	1.3599	1.2942	1.2565
		Present	1.9385	1.6504	1.4914	1.3599	1.2942	1.2565

Table 3. *Variation of non-dimensional fundamental natural frequencies λ of (S-S) FG beams with respect to η, k (K_w =10)*

k	L/h	Theory	$\bar{P}_{cr}$	η						
				0 (linear)	-0.6	-0.8	-0.9	-1.1	-1.2	-1.4
0.2	5	FOBT	52.1649	5.4842	3.4689	2.4531	1.7348	1.7341	2.4526	3.4688
		EBT	56.0561	5.6697	3.5860	2.5357	1.7930	1.7930	2.5358	3.5862
	10	FOBT	55.0355	5.6883	3.5978	2.5442	1.7992	1.7984	2.5436	3.5974
		EBT	56.0784	5.7408	3.6308	2.5674	1.8154	1.8154	2.5674	3.6308
	20	FOBT	55.8189	5.7453	3.6339	2.5697	1.8173	1.8164	2.5691	3.6334
		EBT	56.0838	5.7589	3.6422	2.5754	1.8211	1.8211	2.5754	3.6422
5	5	FOBT	27.8813	4.5107	2.8562	2.0203	1.4288	1.4293	2.0216	2.8599
		EBT	29.4974	4.6179	2.9246	2.0688	1.4632	1.4637	2.0704	2.9290
	10	FOBT	29.2650	4.6978	2.9713	2.1011	1.4857	1.4858	2.1012	2.9716
		EBT	29.7129	4.7321	2.9931	2.1165	1.4966	1.4966	2.1166	2.9933
	20	FOBT	29.6496	4.7498	3.0040	2.1242	1.5020	1.5020	2.1242	3.0040
		EBT	29.7644	4.7589	3.0098	2.1282	1.5049	1.5049	2.1282	3.0098

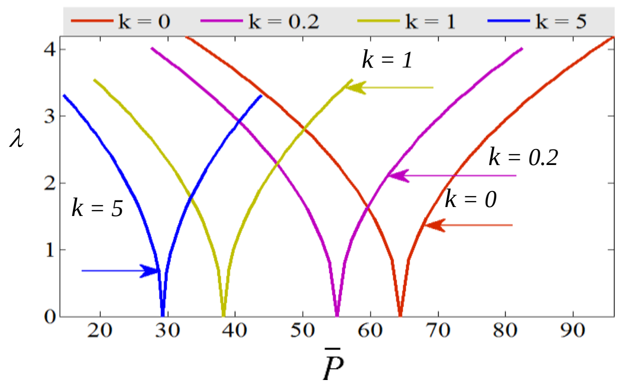

Figure 2. *Variation of the non-dimensional fundamental natural frequencies λ of FG Timoshenko beams around the buckling domain with different values power index k (L/h = 10, K_w = 10, S-S beam)*

Figure 2 displays the effects of material property distributions through the value of the power-law exponent *k* on the frequency of the beam around the buckling domain. When *k* increases, the position of the buckling domain shifts to the left (critical buckling load decreases). This is because the reduced density of the Al_2O_3 constituent leads to the stiffness reduction of the beam. It can be seen that *k* has significant effects on the frequency of the beam.

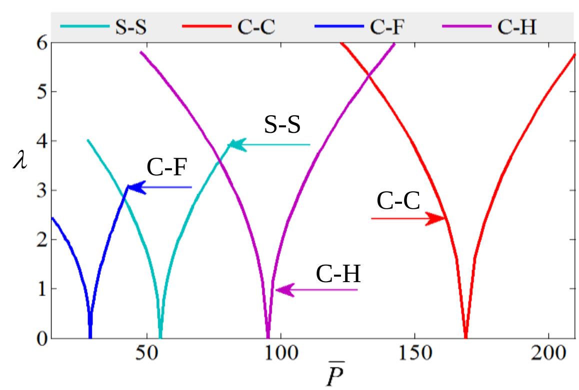

Figure 3. *Variation of the non-dimensional fundamental natural frequencies λ of FG Timoshenko beams around the buckling domain with different boundary conditions (L/h = 10, K_w = 10, k = 0.2)*

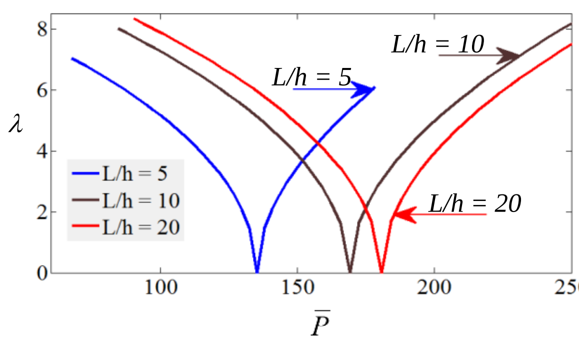

Figure 4. *Variation of the non-dimensional fundamental natural frequencies of FG Timoshenko beams around the buckling domain with different L/h ratios (K_w = 10, k = 0.2, C-C beam)*

Similarly, the effects of boundary conditions on the natural frequency and the buckling domain are depicted in Figure 3; the C-C beam has the highest critical buckling load.

Figure 4 plots the variation of the fundamental frequency of C-C beam with respect to *L/h* ratios. The different *L/h* ratios give different results. It is due to--shear deformation effect that can be observed in Figure 5 for *two cases of beam (EBT and FOBT beam).*

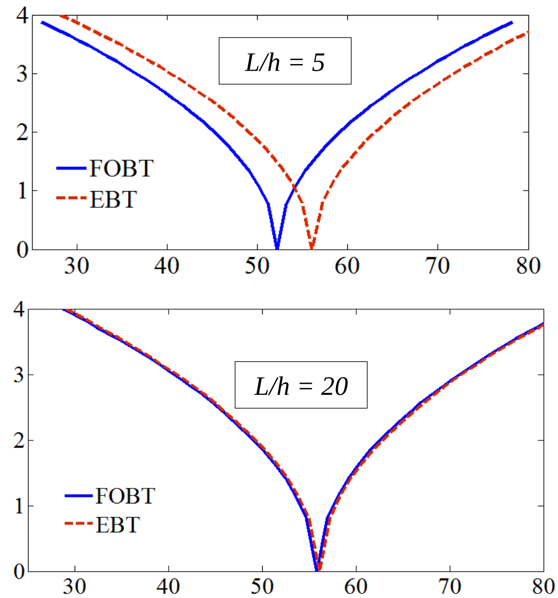

Figure 5. *Variation of the non-dimensional fundamental natural frequencies λ of FG Euler and Timoshenko beams around the buckling domain with different L/h ratios (K_w = 10, k = 0.2, S-S beam)*

4. Conclusions

The free vibration of FG beams on the Winkler foundation around the buckling domain is analyzed with the aid of FEM. Euler-Bernoulli and Timoshenko beam theories are used to consider and evaluate the shear deformation effect on dynamic responses. The developed theory and proposed solution are reliable. Some important conclusions can be drawn from the results of numerical analysis:

• Natural frequency has large variation according to the change in the value of compressive force around buckling domains.

• The closer the compressive force is to the position of critical buckling load, the faster the speed of variation of frequency value is, and the frequency values are almost symmetric with respect to this position.

• Critical buckling loads can also be determined from the condition of zero frequencies.

• Boundary conditions, material distributions, shear deformation and *L/h* ratios also have significant effects on the position of buckling regions.

REFERENCES

[1] Chu Quốc Thắng, *Phương pháp phần tử hữu hạn*, Nhà xuất bản khoa học kỹ thuật, Hà Nội, 1997.

[2] Đào Huy Bích, Nguyễn Đăng Bích, *Cơ học môi trường liên tục*, Nhà xuất bản xây dựng, Hà Nội, 2002

[3] A. Shahba, S. Rajasekaran. "Free vibration and stability of tapered Euler–Bernoulli beams made of axially functionally graded materials", *Applied Mathematical Modelling*, 36, 2012, pp. 3094–3111.

[4] A. Shahba, R. Attarnejad, M. Tavanaie Marvi, S. Hajilar. "Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions", *Composites: Part B*, 42, 2011, pp. 801–808.

[5] A. Fallah, M.M. Aghdam. "Nonlinear free vibration and post-buckling analysis of functionally graded beams on nonlinear elastic foundation", *European Journal of Mechanics A/Solids*, 30, 2011, pp. 571-583

[6] A.S. Kanani, H. Niknam, A.R. Ohadi, M.M. Aghdam. "Effect of nonlinear elastic foundation on large amplitude free and forced vibration of functionally graded beam", *Composite Structures*, 115, 2014, pp. 60-68.

[7] G.H. Rahimi, M.S. Gazor, M. Hemmatnezhad, H. Toorani. "On the post-buckling and free vibrations of FG Timoshenko beams", *Composite Structures*, 95, 2013, pp. 247–253.

[8] H.-T. Thai, T.P. Vo, "Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories", *International Journal of Mechanical Sciences*, 62, 2012, pp. 57–66.

[9] H.-S. Shen. *Functionally graded materials: nonlinear analysis of plates and shells*. Published by Taylor & Francis Group, LLC, 2009.

[10] K.K. Pradhan, S. Chakraverty. "Effects of different shear deformation theories on free vibration of functionally graded beams", *International Journal of Mechanical Sciences*, 82, 2014, pp. 149–160.

[11] M. Aydogdu, V. Taskin. "Free vibration analysis of functionally graded beams with simply supported edges", *Material and Design*, 28, 2007, pp. 1651–1656.

[12] M. Simsek. "Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories", *Nuclear Engineering and Design*, 240, 2010, pp. 697–705.

[13] N. Wattanasakulpong, B. Gangadhara Prusty, D. W.Kelly. "Thermal buckling and elastic vibration of third-order shear deformable functionally graded beams", *International Journal of Mechanical Sciences*, 53, 2011, pp. 734–743

[14] P. Tossapanon, N. Wattanasakulpong N. "Stability and free vibration of functionally graded sandwich beams resting on two-parameter elastic foundation", *Composite Structures*, 142, 2016, pp. 215–225.

[15] S. Kumar, A. Mitra, H. Roy. "Geometrically nonlinear free vibration analysis of axially functionally graded taper beams", *Engineering Science and Technology, an International Journal*, 18, 2015, pp. 579-593.

[16] S-R. Li, RC. Batra. "Relations between buckling loads of functionally graded Timoshenko and homogeneous Euler–Bernoulli beams". *Composite Structures*, 95, 2013, pp.5–9

[17] T-P.Vo, H-T. Thai, T-K.Nguyen, M. Alireza, J. Lee. "Finite element model for vibration and buckling of functionally graded sandwich beams based on a refined shear deformation theory" *Engineering Structures*, 64, 2014, pp. 12–22.

[18] T-K. Nguyen, T-P. Vo, H-T. Thai. "Static and free vibration of axially loaded functionally graded beams based on the first-order shear deformation theory", *Composites: Part B*, 55, 2013, pp. 147–157.

(The Board of Editors received the paper on 05/09/2017, its review was completed on 23/10/2017)