A COMPARATIVE ANALYSIS OF PASSIVITY-BASED CONTROL APPROACHES WITH APPLICATION TO LINEAR DYNAMICAL SYSTEMS

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Abstract - This study focuses on linear dynamical systems whose dynamics are affine in the control input. Such dynamics are extensively considered to be rewritten into a canonical form, namely the passive port-Hamiltonian representation in order to further explore some structural properties such as interconnection and damping matrices, Hamiltonian storage function and dissipation term. On this basis, the passivity-based control design approaches including proportional controller and energy shaping controller are proposed for the purpose of stabilization. Interestingly, the energy shaping controller seems to be better since the controller gain accepts a larger domain of validity and can even be negative. A mass-spring-damper system is used to illustrate the proposed approach. Besides, numerical simulations are included in both the open loop and closed loop to compare the results.

Key words - Port-Hamiltonian representation; modeling; passivity-based control; proportional controller; energy shaping controller.

1. Introduction

This paper deals with the port-based modeling of general nonlinear dynamical systems [1–3] whose dynamics are described by a set of Ordinary Differential Equations (ODEs) and affine in the input *u* as follows:

$$\frac{dx}{dt} = f(x) + g(x)u; \;\; x(t=0) = x_{init} \tag{1}$$

where $x = x(t)$ is the state vector in the operating region $D \subset \mathbb{R}^n$; $f(x) \in \mathbb{R}^n$ expresses the smooth (nonlinear) function with respect to the vector field x. The input-state map and the control input are represented by $g(x) \in \mathbb{R}^{nxm}$ and $u \in \mathbb{R}^m$, respectively. It is worth noting that many industrial applications of electrical, mechanical or biochemical engineering belong to this kind of systems [4–7].

Many control methodologies have been developed for the stabilization of the system (1) at a desired set-point x^*. This is for instance the case for sliding mode control, adaptive control and model predictive control, etc. to cite a few. Recently, passivity-based control (PBC) methodology which is recognized as an extension of Lyapunov approach has attracted much attention from researchers and practitioners. In the PBC framework, it is always important to transfer the (original) dynamics (1) to the port-Hamiltonian (pH) representation prior to developing state feedback laws for control [6, 8]. The control design and control scenarios proposed for the simulations are main contributions of this work.

This paper is organized as follows. Section 2 gives a brief overview of the pH representation of (affine) dynamical systems. Section 3 is devoted to the case study of a mass-spring-damper system modeled within pH framework. Passivity-based control designs (including proportional controller and energy shaping controller) and comparative simulations are given in Section 4. Section 5 ends the paper with some concluding remarks.

Notations: The following notations are considered throughout the paper:

- $\mathbb{R}$ is the the set of real number.
- T is the matrix transpose operator.
- m and n ($m \le n$) are the positive integers.
- x^* is the set-point.
- x_{init} is the initial value of the state vector.

2. The passive port-Hamiltonian (pH) representation

Assume that if the function $f(x)$ verifies the so-called separability condition [7, 9], that is, $f(x)$ can be decomposed and expressed as the product of some (interconnection and damping) structure matrices and the gradient of a potential function with respect to the state variables, i.e., the co-state variables:

$$f(x) = \left[J(x) - R(x)\right]\frac{\partial H(x)}{\partial x} \tag{2}$$

where $J(x)$ and $R(x)$ are the $n \times n$ skew-symmetric interconnection matrix (i.e., $J(x) = -J(x)^{\mathrm{T}}$) and the $n \times n$ symmetric damping matrix (i.e., $R(x) = R(x)^{\mathrm{T}}$), respectively while $H(x): \mathbb{R}^n \to \mathbb{R}$ represents the Hamiltonian storage function of the system (possibly related to the total energy of the system). Furthermore, if the damping matrix $R(x)$ is positive semi-definite,

$$R(x) \ge 0 \tag{3}$$

Then, the original dynamics described by (1) is said to be a pH representation with dissipation [4, 5]. Equation (1) is then rewritten as follows:

$$\begin{cases} \dot{x} = \left[J(x) - R(x)\right]\dfrac{\partial H(x)}{\partial x} + g(x)u \\ y = g(x)^{\mathrm{T}}\dfrac{\partial H(x)}{\partial x} \end{cases} \tag{4}$$

where y is the output.

It can be clearly seen for the pH models defined by (3) (4) that the time derivative of the Hamiltonian storage function $H(x)$ satisfies the energy balance equation [5] below:

$$\frac{dH(x)}{dt} = -\left[\frac{\partial H(x)}{\partial x}\right]^{\mathrm{T}} R(x)\frac{\partial H(x)}{\partial x} + u^{\mathrm{T}} y \tag{5}$$

Thanks to (3), (5) becomes:

$$\underbrace{\frac{dH(x)}{dt}}_{\text{stored power}} \leq \underbrace{u^T y}_{\text{supplied power}} \quad (6)$$

From a physical point of view, inequality in (6) implies that the total amount of energy supplied from external source is always greater than the increase in the energy stored in the system. Also, equality in (6) holds only if the damping matrix $R(x)$, which is strongly related to the dissipation term, is equal to 0. Hence, the pH system (4) is said to be passive with input u and output y corresponding to the Hamiltonian storage function $H(x)$ [3]. This is one of advantageous features of the pH representation and has been applied to the control design, even for the stabilization of infinite dimensional systems (see, e.g. [10, 11]).

We shall not elaborate any further on the pH representation here (for example, the concepts related to the cyclopassive/ passive property or Dirac structure, etc.) and refer the reader to [4, 6, 9] for more information.

3. A case study: Mass-spring-damper system

To illustrate the concepts proposed in Section 2, we illustrate our main points with a simple case study, which is the mass-spring-damper system. Originally, the port Hamiltonian representation has been first considered for electrical or mechanical systems as seen in the literature (see, e.g. [2, 12]).

Buildings or suspension structure of a vehicle traveling over a bumpy road can be modeled as a mass-spring-damper system in a vertical position[1] as shown in Figure 1 [12].

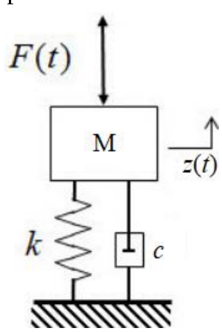

Figure 1. *A mass-spring-damper system*

The following equation is derived using Newton's second law [14][2]:

$$M\frac{d^2z(t)}{dt^2} = F - kz(t) - c\frac{dz(t)}{dt} \quad (7)$$

where:

- M is the mass of the body;
- F is the external force;
- k is the stiffness constant of the (linear) spring;
- c is the damping constant.

Let x be the vector consisting of the position $z(t)$ and the momentum of the body $M\frac{dz(t)}{dt}$, i.e. $x = (x_1, x_2)^{\mathrm{T}} \equiv \left(z(t), M\frac{dz(t)}{dt}\right)^{\mathrm{T}}$, (7) can be rewritten as follows:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \begin{pmatrix} kx_1 \\ \frac{x_2}{M} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F \quad (8)$$

The system dynamics (8) lead to a pH representation (4) with:

$$J(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (9)$$

$$R(x) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \quad (10)$$

$$g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u = F \quad (11)$$

$$y = \frac{x_2}{M} \equiv \frac{dz(t)}{dt} \text{ (the velocity)} \quad (12)$$

and, $$H(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}\frac{x_2^2}{M} \quad (13)$$

In this case, the Hamiltonian storage function $H(x)$ (13) is equal to the total energy of the system, (i.e., it characterizes the amount of the elastic potential energy of the spring and the kinetic energy of the body, respectively). It therefore has the unit of energy. The damping matrix $R(x)$ (10) is symmetric and positive semi-definite.

Remark 1. The system states x_1 and x_2 converge to the nonzero values at steady state (i.e., $x_1^* = \frac{F}{k}$ and $x_2^* = 0$) if F is different from 0.

Remark 2. As an analogy between mechanical and electrical systems [15], it is worth noting that a second order ordinary differential equation of the series RLC circuit operated under a voltage source $V(t)$ can be written as follows:

$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dV(t)}{dt}$ where $i(t)$ is the electric current. This is clearly equivalent to (7) in some sense.

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[1] Fixed-base or base-excited configuration [13] can be handled similarly.

[2] This belongs to the so-called (generalized) Euler-Lagrange equations of classical mechanics [2]. Equation (7) is still valid even in the presence of friction where the friction force is assumed to be proportional to the velocity.

4. Passivity-based control design and simulations

4.1. The proportional controller design

Let us state the following proposition.

Proposition 1. Under a zero state detectability condition[3] and the boundedness from below of the Hamiltonian storage function $H(x)$ by 0, it follows that an explicit proportional static output feedback law of the form

$$u = -K_p y \tag{14}$$

with y given by (12) and $K_p > 0$ a so-called damping injection gain, renders the controlled pH system (4) with (9)–(13) dissipative and therefore asymptotically stabilized at the (singular) equilibrium $x^* = (0,0)^{\mathrm{T}}$ [4].

Proof. From (6) and (14), one obtains:

$$\frac{dH(x)}{dt} \le y^{\mathrm{T}} K_p y < 0$$

The proof is followed immediately by invoking La Salle's invariance principle [1, 6]. A complete version of the proof can be found in [11].

Remark 3. The convergence speed of the controlled system goes faster by increasing the controller gain K_p. Better performance of the controller can be proposed with the gain K_p which is derived from the Ziegler-Nichols tuning method.

4.2. The energy shaping controller design

Proposition 2. A state feedback law

$$u = -\frac{1}{K}x_1 + \left(k + \frac{1}{K}\right)x_1^* \tag{15}$$

with $\left(k + \frac{1}{K}\right) > 0$ asymptotically stabilizes the system (4) with (9)–(13) at the (nonsingular) equilibrium $x^* = (x_1^*, 0)^{\mathrm{T}}$.

Proof. From (15), we consider $u \triangleq -\frac{dH_a(x_1)}{dx_1}$ which leads to $H_a(x_1) = \frac{1}{2K}x_1^2 - \left(k + \frac{1}{K}\right)x_1^* x_1$. On the other hand, let $H_d(x)$ be the (closed-loop) Hamiltonian storage function, i.e., $H_d(x) = H(x) + H_a(x_1)$ [5]. We can easily check that the function $H_d(x)$ admits a global minimum at $x^* = (x_1^*, 0)$ since its Hessian matrix $\begin{pmatrix} \left(k + \frac{1}{K}\right) & 0 \\ 0 & \frac{1}{M} \end{pmatrix} > 0$ and the time derivative $\frac{dH_d(x)}{dt} = -c\left(\frac{x_2}{M}\right)^2 < 0$. The latter completes the proof.

4.3. Numerical simulations

The simulations are carried out using MATLAB & SIMULINK. The model parameters are given with $k = 0.25$ (N/m), $c = 0.5$ (N/(m.s)) and $M = 6.25$ (kg) (see also [13])[6]. The input force imposed on the system is a unit step, i.e., $u(t) = S(t)$ where $S(t)$ is the unit step function. The initial conditions are chosen to be $x_1(t=0) = x_{1,init} = 3$ and $x_2(t=0) = x_{2,init} = 0$. Figure 2 shows the time evolutions of the states and the storage function. It is shown that the storage function is bounded from below by a positive scalar since steady states are different from 0 (see (13) and *Remark* 1).

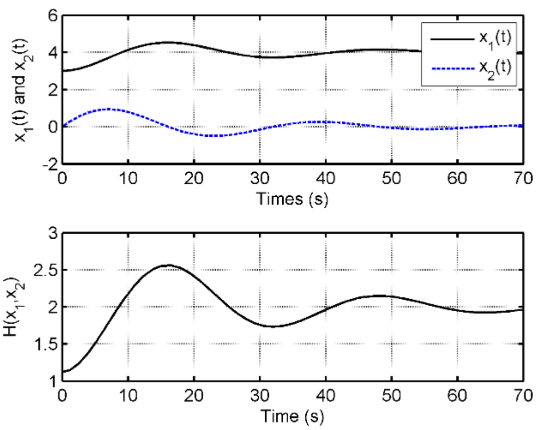

Figure 2. *The states and storage function w.r.t. time*

In what follows, without loss of generality we propose to stabilize the system at the natural equilibrium $x^* = (0,0)^{\mathrm{T}}$. The closed loop simulations with the proportional feedback law (14) are implemented with $K_p = 0.1$ which is derived from the Ziegler-Nichols tuning method. On the other hand, the stability condition for the gain K of the energy shaping controller (15) is either $K > 0$ or $K < -\frac{1}{k} = -4$. Let us choose $K = -10$ so that $K_p = \frac{1}{|K|}$ for the purpose of comparisons. Since the controlled Hamiltonian storage functions converge to 0 as $t \to +\infty$ (Figure 3), it implies that the system (4) with (9)–(13) is globally stabilized with (14) and (15) (see the closed loop phase plane in Figure 4). In both cases, the global convergence of the controlled states x to x^* is

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[3] This condition is a version weaker than the observability condition.

[4] If a nonzero equilibrium $x^* = (x_1^*, 0)$ is considered as desired set-point, the proposed result can also be deduced similarly using a coordinate transformation given by $\bar{x} = x - x^*$.

[5] Hence, the open loop Hamiltonian storage function has been shaped by $H_a(x_1)$ to become $H_d(x)$.

[6] It can be shown that the damping factor $\zeta \triangleq \frac{1}{2}\frac{c}{\sqrt{kM}}$ equals 0.2. The open loop system is therefore underdamped.

guaranteed. However, the exponential spiral orbit via (15) is embedded by the orbit generated by (14) (i.e., it seems to be better for the practical implementations). In addition, the manipulated inputs u given by (14) and (15) are physically admissible in terms of amplitude and dynamics as seen in Figure 5.

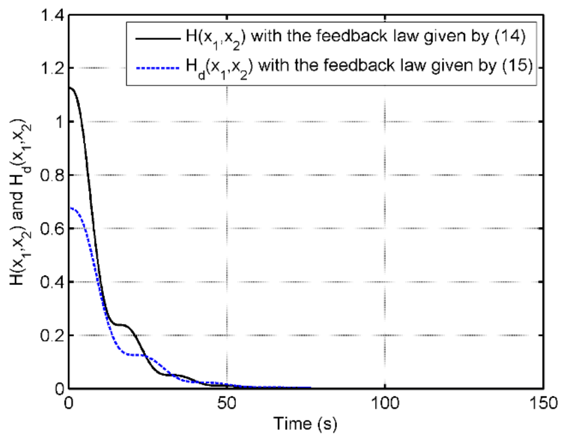

Figure 3. *The controlled Hamiltonian storage function*

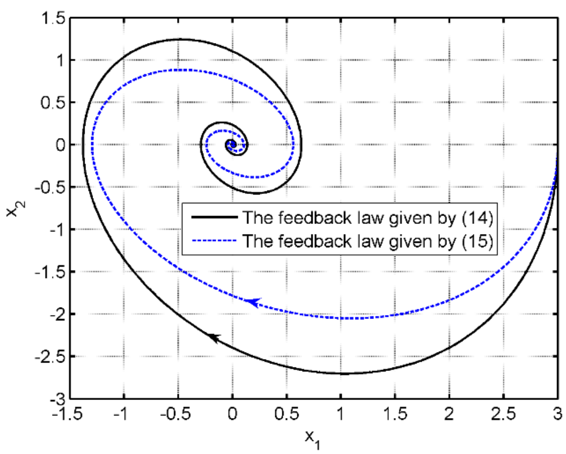

Figure 4. *The closed loop phase plane*

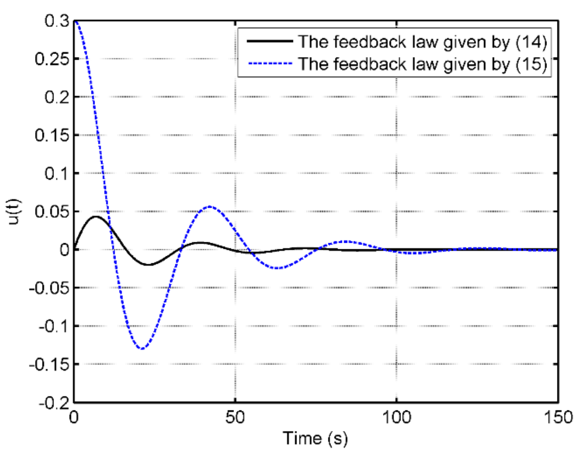

Figure 5. *The manipulated input w.r.t. time*

5. Conclusion

In this paper, a mass-spring-damper system is used to introduce the so-called port-Hamiltonian representation. In this presentation, some structural properties such as interconnection and damping matrices, Hamiltonian storage function and dissipation term are highlighted from a physics point of view. The feedback designs (including proportional controller and energy shaping controller) and control scenarios proposed for the comparisons are main contributions of the paper. Interestingly, the energy shaping controller seems to be better for the practical implementations since the controller gain accepts a larger domain of validity and can even be negative. It remains now to adapt the proposed results to nonlinear multiphysics multiscale systems.

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