NUMERICAL SOLUTIONS OF THE DIFFUSION COEFFICIENT IDENTIFICATION PROBLEM

NGHIỆM SỐ CHO BÀI TOÁN XÁC ĐỊNH HỆ SỐ TÁN XẠ

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Abstract - In this paper, we investigate several numerical algorithms to find the numerical solutions of the diffusion coefficient identification problem. Normally, in order to solve this problem, one uses the least squares function together with a regularization method, butwe here use the energy functional with parsity regularization method. Our approach leads to the study of a minimum convexproblem (but not differentiable). Therefore, we can apply some fast and efficient algorithms, which has been proposed recently. The main results presented in the paper is to give the new approach and to implement the efficient algorithms to find the numerical solutions of the diffusion coefficient identification problem. The effectiveness of the algorithms and the numerical solutions are illustrated and presented in a specific example.

Key words - sparsity regularization; energy functional; diffusion coefficient identification problem; Gradient-type algorithm; Nesterov's accelerated algorithm; Beck's accelerated algorithms; numerical solution.

Tóm tắt - Trong bài báo này, chúng tôi nghiên cứu một số giải thuật để tìm nghiệm số cho bài toán xác định hệ số khuếch tán. Thông thường, để giải bài toán này, người ta dùng hàm bình phương tối thiểu được chỉnh hóa nhưng ở đây, chúng tôi dùng phiếm hàm năng lượng cùng với phương pháp chỉnh hóa thưa. Cách tiếp cận của chúng tôi dẫn đến việc nghiên cứu một bài toán cực tiểu lồi (nhưng không trơn). Vì thế chúng tôi có thể áp dụng được các giải thuật nhanh và hiệu quả, mà đã được đưa ra gần đây. Kết quả chủ yếu của bài báo thể hiện ở cách tiếp cận mới và việc ứng dụng các giải thuật để tìm nghiệm số của bài toán xác định hệ số khuếch tán. Tính hữu hiệu của giải thuật và các nghiệm số được ứng dụng và minh họa trong một ví dụ số cụ thể.

Từ khóa - chỉnh hóa thưa; phiếm hàm năng lượng; bài toán xác định hệ số khuếch tán; phương pháp kiểu Gradient; phương pháp tăng tốc của Nesterov; phương pháp tăng tốc của Beck; nghiệm số.

1. Introduction

The diffusion coefficient identification problem is to identify the coefficient σ in the equation

$$-\operatorname{div}(\sigma\nabla\varphi) = y \text{ in } \Omega, \varphi = 0 \text{ on } \partial\Omega \quad (1)$$

from noisy data $\varphi^\delta \in H_0^1(\Omega)$ of φ.

It is well-known that the problem is ill-posed and thus need to be regularized. There have been several regularization methods proposed. Among of them, Tikhonov regularization [5,3] and the total variational regularization [10,2] are most popular. The numerical solutions of the problem have also examined. However, their quality has not been satisfaction yet. For surveys on this problem, we refer to [5] and the references therein.

2. Solutions

One way to improve the quality of approximations is to use prior information of the solution of the problem as much as possible. In some applications, the coefficient σ^*, which needs to be recovered, has a sparse presentation, i.e. the number of nonzero components of $\sigma^* - \sigma^0$ are finite in an orthonormal basis (or frame) of $L^2(\Omega)$. In fact, we assume that σ^* belongs to the set A defined by

$$A = \{\sigma \in L^\infty(\Omega) : \sigma \in [\lambda, \lambda^{-1}] \quad (2)$$

$$\text{and } \operatorname{supp}(\sigma - \sigma^0) \subset \Omega' \subset\subset \Omega\},$$

where Ω' is an open set with the smooth boundary that contained compactly in Ω, the constant $\lambda \in (0,1)$ and σ^0 is the background value of σ that has already known.

The sparsity of $\sigma^* - \sigma^0$ promotes to use sparsity regularization since the method is simple for use and very efficient for inverse problems with sparse solutions. This method has been of interest by many researchers for the last years. For nonlinear inverse problems, the well-posedness and convergence rates of the method have been analyzed, e.g. [4]. Some numerical algorithms have also been proposed, e.g. [7].

Here, instead of the approach in [4] we use the energy functional approach incorporating with sparsity regularization, i.e. considering the minimization problem

$$\min_{\sigma\in A} F_{\varphi^\delta}(\sigma) + \alpha\Phi(\sigma - \sigma^0), \quad (3)$$

where A is an admissible set defined by (2) and $\alpha > 0$ is a regularization parameter, and

$$F_{\varphi^\delta}(\sigma) := \int_\Omega \sigma \left|\nabla\left(F_D(\sigma)y - \varphi^\delta\right)\right|^2 dx, \quad (4)$$

$$\Phi(\vartheta) := \sum \omega_k \left|\langle \vartheta, \phi_k \rangle\right|^p, \quad (1 \le p \le 2) \quad (5)$$

with $F_D(\cdot)y : A \to H_0^1(\Omega)$ mapping the coefficient $\sigma \in A$ to the solution $u = F_D(\sigma)y$ of problem (1), $\{\phi_k\}$ being an orthonormal basis (or frame) of $L^2(\Omega)$ and $\omega_k \ge \omega_{min} > 0$ for all k. Note that for $p = 1$, the minimizers of (3) is sparse and thus the method is suitable with the setting of our problem.

The advantage of our approach is to deal with a convex problem. Therefore, its global minimizers are easy to find and some efficient algorithms for convex problems can be applied [7]. Moreover, as shown in [8], the well-posedness of problem (3) is obtained without further condition and the source condition of the convergence rates is very simple.

Note that the energy functional approach was used by several researchers such as Zou [10], Knowles [6] and Hao and Quyen [5].

3. Study Results and Comments

3.1. Notations

We recall that a function φ in $H_0^1(\Omega)$ is a weak solution of (1) if the identity

$$\int_\Omega \sigma\nabla\varphi\cdot\nabla v dx = \int_\Omega y v dx \tag{6}$$

holds for all $v\in H_0^1(\Omega)$. If $\sigma\in A$ and $y\in L^2(\Omega)$, then there is an unique weak solution $\varphi\in H_0^1(\Omega)$ of (1) [5].

We now assume that φ^* is an exact solution of problem (1), i.e. there exists some $\sigma^*\in A$ such that $\varphi^* = F_D(\sigma^*)y$, and only noisy data $\varphi^\delta\in H_0^1(\Omega)$ of φ^* such that

$$\left\|\varphi^* - \varphi^\delta\right\|_{H^1(\Omega)} \le \delta$$

with $\delta > 0$ are given. As concerned, sparsity regularization incorporated with the energy functional approach leads to considering the minimization problem

$$\min_{\sigma\in L^2(\Omega)} \Theta(\sigma) := F_{\varphi^\delta}(\sigma) + \alpha\Phi(\sigma - \sigma^0), \tag{7}$$

where F_{φ^δ} and Φ are given by (4) and (5), respectively. Here, $\Theta(\sigma)$ is set to be infinity if σ is not belong to $A\cap dom(\Phi)$.

Note that since the functionals $F_{\varphi^\delta}(\cdot)$ and $\Phi(\cdot)$ are convex, the minimization problem (7) is convex. Therefore, we can use some efficient algorithms to solve it. In this paper, we aim at presenting some fast algorithms for minimization problem (7). For simplicity, we present the algorithms for the minimization problem

$$\min_{u\in H} \Theta(u) := F(u) + \Phi(u), \tag{8}$$

where $F(\cdot): H\to R$ is a Fréchet differentiable functional and $\Phi(u)$ is defined by $\Phi(u) = \alpha\sum_{k\in\Lambda}\omega_k|\langle u,\varphi_k\rangle|^p$ with $p\in[1,2]$ and $\{\phi_k\}$ is an orthonormal basis (or frame) of Hilbert space H. The problem (3) is a case of the problem (8).

3.2. Differentiability

In order to present algorithms, the differentiability of the operator $F_\varphi(\cdot)$ is needed, which is obtained in the following theorem:

Theorem 1.*[8]* For $\varphi\in H_0^1(\Omega)$, the functional $F_\varphi(\cdot): A\subset L^q(\Omega)\to R$ defined by

$$F_\varphi(\sigma) = \int_\Omega \sigma\left|\nabla\left(F_D(\sigma)y - \varphi\right)\right|^2 dx$$

has the following properties

1) For $q\ge 1, \frac{1}{q}+\frac{1}{r}=\frac{1}{2}$ and $y\in L^r(\Omega)$, $F_\varphi(\cdot)$ is continuous with respect to the L^q – norm.

2) For $y\in L^{r+\varepsilon}(\Omega)$ with $\varepsilon>0$, there exists $q>2$ such that $F_\varphi(\cdot)$ is Fréchet differentiable with respect to the L^q -norm and

$$F'_\varphi(\sigma)\vartheta = -\int_\Omega \vartheta\left(\left|\nabla F_D(\sigma)y\right|^2 - \left|\nabla\varphi\right|^2\right)dx.$$

Furthermore, $F_\varphi(\cdot)$ is convex on the convex set A and $F''_\varphi(\cdot)$ is uniformly bounded.

3.3. Algorithms

To solve this problem, there have been several algorithms proposed in [7]. Their convergence have been obtained under different conditions. In the following, we briefly present these algorithms. They consist of the gradient-type algorithm, Beck's accelerated algorithm and Nesterov's accelerated algorithm.

The main idea of the gradient-type method is to approximate the problem (8) by a sequence of minimization problem, $\min_{v\in H}\Theta_{s^n}(v,u^n)$, in which $\Theta_{s^n}(\cdot,u^n)$ are strictly convex and the minimization problems are easy to solve. Furthermore, the sequence of minimizer $u^{n+1} = \text{argmin}_{v\in H}\Theta_{s^n}(v,u^n)$ should converge to a minimizer of problem (8). To this end, the functional $\Theta_{s^n}(\cdot,u^n)$ is chosen by

$$\Theta_{s^n}(v,u^n) := F(u^n) + \langle F'(u^n), v-u^n\rangle + \frac{s^n}{2}\|v-u^n\|^2 + \Phi(v). \tag{9}$$

The functional is strictly convex and has a unique minimizer given by

$$u^{n+1} = S_{\frac{\alpha\omega}{s^n},p}\left(u^n - \frac{1}{s^n}F'(u^n)\right), \tag{10}$$

where $S_{\frac{\alpha\omega}{s^n},p}(\cdot)$ is the soft shrinkage operator defined by

$$\mathbb{S}_{\omega,p}(u) := \sum_{k\in\Lambda} S_{\omega_k,p}(\langle u,\varphi_k\rangle)\varphi_k, \tag{11}$$

with the *shrinkage functions* $S_{\tau,p}: R\to R$ as follows

$$S_{\tau,p}(x) = \begin{cases}\text{sgn}(x)\max(|x|-\tau,0) : p=1\\ G^{-1}_{\tau,p}(x) : p\in(1,2]\end{cases} \tag{12}$$

and

$$G_{\tau,p}(x) = x + \tau p\,\text{sgn}(x)|x|^{p-1} \text{ for } 1<p\leqslant 2. \tag{13}$$

The basis condition of the convergence of the iteration (10) is that in each iterate, the parameter s^n has to be chosen such that

$$\Theta(u^{n+1}) \le \Theta_{s^n}(u^{n+1},u^n).$$

This condition is automatic satisfied when $s^n\ge L$ with L being the Lipschitz constant of F'. The detail of the gradient-type method with a step size control is presented by Alg.1 in [7]. Although the gradient-type algorithm converges for the problem (8) with non-convex functional F, its convergence is very slow. Its order of the convergence is $O(1/n)$. For the minimization problem of our interest, the functional F is convex. Therefore, we can use the more efficient algorithms in [7,1,9], Beck's accelerated algorithm and Nesterov's accelerated algorithm. These algorithms converge with the order of convergence $O(1/n^2)$, which is known to be thebest for the algorithms using only the gradient and values of the objective functional [7].

The main idea of Beck's accelerated algorithm is to construct two sequences $\{u^n\}$ and $\{y^n\}$:

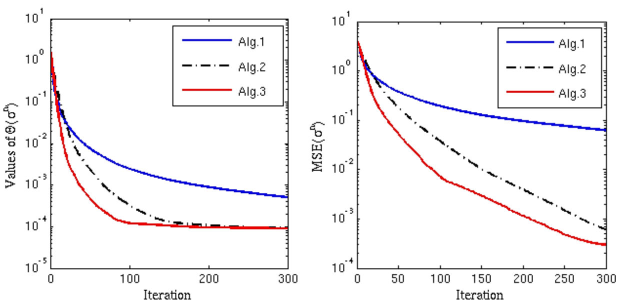

Figure 1. *Values of* $\Theta(\sigma^n), MSE(\sigma^n)$ *and step size* $1/s^n$ *in Alg.1, Alg.2 and Alg.3 in the case of free noise*

1. $y^n = u^n + t_n(u^n - u^{n-1})$
2. $u^{n+1} = S_{\frac{\alpha\omega}{s^n},p}\left(y^n - \frac{1}{s^n}F'(y^n)\right),$

and together with clever choice of parameters t_n and s^n, the convergence rate of the algorithm is of order $O(1/n^2)$. The detail of this algorithm is given by Alg.2 in [7].

In Nesterov's accelerated algorithm, they construct three sequences $\{u^n\},\{y^n\}$ and $\{v^n\}$:

1. $v^n = S_{\alpha\omega A_n,p}\left(u^0 - \sum_{k=1}^{n} a_k F'(u^k)\right)$
2. $y^n = t_n u^n + (1-t_n)v^n$
3. $u^{n+1} = S_{\frac{\alpha\omega}{s^n},p}\left(y^n - \frac{1}{s^n}F'(y^n)\right).$

Together with specific choices of parameters a_n, A_n, t_n and s^n, the algorithm converges with the order of convergence $O(1/n^2)$. The detail of the algorithm is presented in Alg.3 in [7].

3.4. Numerical solutions

For illustrating the algorithms, we assume that Ω is the unit disk and

$$\sigma^*(x_1,x_2) = \begin{cases} 3, (x_1,x_2)\in B_1 \\ 4, (x_1,x_2)\in B_2 \\ 3, (x_1,x_2)\in R \\ 1, otherwise \end{cases}, \; y(x_1,x_2) = 4\sigma^*,$$

where

$$B_1 = B_{0.3}(-0.4, 0.4),\; B_2 = B_{0.3}(-0.4,-0.4),$$
$$R = [0.2, 0.5]\times[-0.3, 0.3]$$

with $B_r(x_1; x_2)$ being the disk with center at $(x_1; x_2)$ and radius r.

To obtain φ^* we solve (1) with $\sigma = \sigma^*$ and $y = 4\sigma^*$ by the finite element method on a mesh with 1272 triangles. The solution of (1) as well as the parameter σ are represented by piecewise linear finite elements. The algorithms above described will compute a sequences σ^n for approximating σ^*. In order to maintain the ellipticity of the operator, we add as usual an additional truncation step in the numerical procedure, which, however, is not covered by our theoretical investigation, i.e. we have cut off values of σ^n which are below $\sigma^0 = 1$ in each iteration.

To obtain $\varphi^\delta \in H_0^1(\Omega)$, we first choose $y^\delta = y + 5\frac{R}{\|R\|_{L^2(\Omega)}}$, where R is computed with the MATLAB routine $randn(size(y))$ with setting $randn('state',0)$. φ^δ is then obtained by solving (1) with y replaced by y^δ. We obtain

$$\|\phi^\delta - \phi^*\|_{\mathbb{H}^1(\Omega)} = 0.0928 \approx 0.1,$$

$$\frac{\|\phi^\delta - \phi^*\|_{\mathbb{H}^1(\Omega)}}{\|\phi^*\|_{\mathbb{H}^1(\Omega)}} = 0.0044.$$

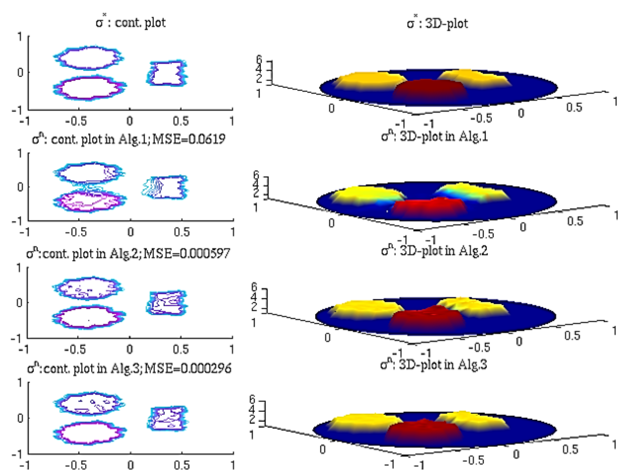

Figure 2. *3D-plots and contour plots of* σ^* *and* $\sigma^n, n = 300$ *in Alg.1, Alg.2 and Alg.3 in the case of free noise*

Using this specific example, we analyze the gradient-type method (Alg.1) and its accelerated versions, Alg.2 and Alg.3 in [7]. In these algorithms, we set

$$p = 1, \mu = 0.5, \omega_k = 1, [\underline{s}, \overline{s}] := [10^{-2}, 10^2], \alpha := 5\cdot 10^{-5}$$

We measure the convergence of the computed minimizers to the true parameter σ^* by considering the mean square error sequence

$$MSE(\sigma^n) = \int_\Omega (\sigma^n - \sigma^*)^2 dx.$$

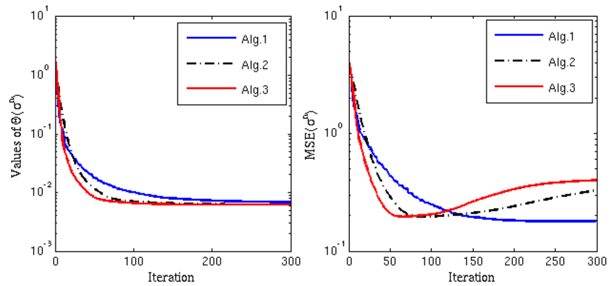

Figure 3. *Values of* $\Theta(\sigma^n), MSE(\sigma^n)$ *and step size* $1/s^n$ *in Alg.1, Alg.2 and Alg.3 in the case of* 10% *noise*

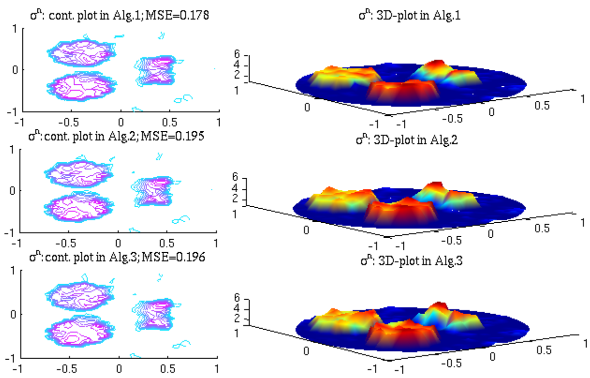

Figure 4. *3D-plots and contour plots of* σ^* *and* σ^n *in Alg.1, Alg.2 and Alg.3 in the case of* 10% *noise. In the algorithms,* n *is taken with respect to the minimum value of* $MSE(\sigma^n)$.

Figures 1 and 3 illustrate the values of $\Theta(\sigma^n)$ and $MSE(\sigma^n)$ in Alg.1, Alg.2 and Alg.3 in two case of data: free noise and 10% noise, respectively. In two cases, the decreasing rate of $\Theta(\sigma^n)$ in two algorithm, Alg.2 and Alg.3, are very rapid and much faster than that in Alg.1. This observation is suitable with the theory result, which the convergence rate of two accelerated algorithms is of order $O(1/n^2)$ and it is $O(1/n)$ for the gradient-type algorithm. Note that although Alg.2 and Alg. 3 have the same order of the convergence rate, Alg.3 converges faster than Alg.2. For the sequence $MSE(\sigma^n)$, an analogous result is also true in the case of free noise. However, in the case of noise data $MSE(\sigma^n)$ decrease in the first iterates, after that they increase. The semi-convergence here is easy to understand since $\{\sigma^n\}$ in three algorithms converge to the minimizer of Θ, which is not papameter σ^*.

Figures 2 and 4 present the plots of σ^* and σ^n in the algorithms with respect to two cases of data, free noise and 10% noise, respectively. They show that σ^n in three algorithms are very good approximations of σ^* in the case of free noise and they are acceptable approximations in the case of noise data.

4. Conclusion

We have investigated the algorithms for sparsity regularization incorporated with the energy functional approach. The advantage of our approach is to work with a convex minimization problem and thus the efficient algorithms can be used. The efficiency of the algorithms has illustrated in a specific example.

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