SELF DUAL SOLUTIONS OF CLASSICAL YANG-MILLS THEORY

NGHIỆM TỰ ĐỐI NGẪU CỦA LÝ THUYẾT YANG-MILLS CỔ ĐIỂN

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Abstract - In this paper, we investigate self dual solutions in Minkowski space of the classical SU(2) Yang-Mills theory. Self dual fields automatically satisfy the motion equations of the pure gauge theory. We present a self dual solution of the pure SU(2) Yang-Mills theory. This solution is imaginary and exhibits to break the local SU(2) symmetry at large distance. From the connection between the pure SU(2) gauge theory and the scalar ϕ^4 theory, we obtain formulas for several interesting quantities: the Yang-Mills field strengths, the Lagrangian density, the energy-momentum tensor. The obtained results prove clearly that if fields are self dual then energy-momentum tensor equals zero. We also find a self dual solution of the corresponding field equation. We see that the spatial component $\left(W_i^a\right)$ of the SU(2) gauge potential is analogous to the potential of a point magnetic monopole.

Key words - self dual solution, classical Yang-Mills theory, equation of motion, gauge potential, SU(2) group.

Tóm tắt - Trong bài báo này chúng tôi khảo sát nghiệm tự đối ngẫu trong không gian Minkowski của lý thuyết Yang-Mills SU(2) cổ điển. Các trường tự đối ngẫu thì tự động thỏa mãn các phương trình chuyển động của lý thuyết chuẩn thuần túy. Chúng tôi đưa ra nghiệm tự đối ngẫu của lý thuyết Yang-Mills SU(2) thuần túy. Nghiệm này là phức và nó biểu lộ sự phá vỡ đối xứng SU(2) định xứ ở khoảng cách lớn. Từ sự liên hệ giữa lý thuyết chuẩn SU(2) thuần túy và lý thuyết vô hướng ϕ^4, chúng tôi nhận được các công thức của một vài đại lượng quan tâm: cường độ trường Yang-Mills, mật độ Lagrangian, tenxơ năng-xung lượng. Các kết quả nhận được đã minh chứng rõ, nếu trường là tự đối ngẫu thì tenxơ năng - xung lượng của trường bằng không. Chúng tôi cũng tìm được nghiệm tự đối ngẫu của phương trình trường tương ứng. Chúng tôi thấy rằng thành phần không gian $\left(W_i^a\right)$ của thế chuẩn SU(2) thì tương tự thế của một monopole từ điểm.

Từ khóa - nghiệm tự đối ngẫu, lý thuyết Yang-Mills cổ điển, phương trình chuyển động, thế chuẩn, nhóm SU(2).

1. Introduction

It is well known that the Yang-Mills theory [1] has become a general framework to formulate theories of fundamental interactions. Solutions of the classical field equations, in which field functions are c-numbers (not operators), play an important role in considering the configurations of the corresponding quantum field theory. Based on these solutions, applying the semi classical analysis methods, some important contents about the quantum theory can be obtained, which could not be done with perturbation theories [2-3].

The classical Yang-Mills equations have interesting solution classes [4-6]: monopole and dyon solutions, instanton self dual solutions, meron solutions, stringlike and vortex solutions,… One important result when studying the classical Yang-Mills equations is to recognize that the extremum of the functional in the Euclidian space does not correspond to the zero-unified field, but corresponds to the nontrivial space-time local field configuration known as instanton. In the quantum theory, instanton describes the tunnel effects of the degenerate vacuum states. This result brought a new view to the vacuum structure of the Yang-Mills theory, providing a qualitative explanation of the quark confinement. The classical Yang-Mills field theory can be studied independently of exact solutions, of course. This is on interesting persuit, because any results gained may lead to improvement in the path integral formulation of quantum Yang-Mills theory [7]. Thus, self dual solutions of the classical Yang-Mills theory have important role in corresponding quantum field theoretical research.

In this paper we study self dual fields, which are solutions of classical Yang-Mills equations with SU(2) gauge group. The paper is organized as follows: In Section 2, we consider interesting properties of self dual fields. Section 3 present a self dual solution of the pure SU(2) Yang-Mills theory. In Section 4, from connection between the Yang-Mills theory with the scalar ϕ^4 theory we investigate another self dual solution of the pure SU(2) Yang-Mills theory. The discussions and conclusions are given in Section 5.

2. Properties of Self Dual Fields

The SU(2) Yang-Mills field in Minkowski space is self dual if it satisfies the condition

$$\tilde{F}_{\mu\nu}^a = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_a^{\alpha\beta} = \pm iF_{\mu\nu}^a, \tag{1}$$

or equivalently,

$$B_n^a = \pm iE_n^a, \tag{2}$$

where

$$E_n^a = F_{0n}^a,\ B_n^a = -\frac{1}{2}\varepsilon_{nmk}F_{mk}^a \tag{3}$$

are the SU(2) electric and magnetic Yang-Mills fields; $F_{\mu\nu}^a$ is Yang-Mills field intensity tensor

$$F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon_{abc}W_\mu^b W_\nu^c. \tag{4}$$

In the above equations: $\mu,\nu,\alpha,\beta = 0,1,2,3$ are space-time indices; $a, b, c, n, m, k = 1,2,3$ are SU(2) group indices and g is the coupled constant.

Self dual fields are interesting because they automatically satisfy the motion equations of the pure gauge theory

$$\partial^{\nu} F^{a}_{\mu\nu} = g\varepsilon_{abc} F^{b}_{\mu\nu} W^{\nu}_{c}. \quad (5)$$

In the SU(2) case, the relation of

$$\partial^{\nu} \tilde{F}^{a}_{\mu\nu} = g\varepsilon_{abc} \tilde{F}^{b}_{\mu\nu} W^{\nu}_{c} \quad (6)$$

is nothing more than an identity. For a self dual field, this identity becomes the equation of motion for the SU(2) gauge theory. Any potential W^{a}_{μ}, which leads to a self dual tensor, is a solution of the motion equation.

One can try to make use of this fact by searching for solutions of the self dual equations, which are the first order, rather than trying to solve the second order equation of motion. Interesting solutions can be found in this way.

Any self dual solution in Minkowski space has a vanishing energy- momentum tensor

$$T_{\mu\nu} = -F^{a}_{\mu\lambda} F^{\lambda}_{a\nu} + \frac{1}{4} g_{\mu\nu} F^{a}_{\alpha\beta} F^{\alpha\beta}_{a}. \quad (7)$$

The components of $T_{\mu\nu}$ are

$$T_{00} = \frac{1}{2}\left(E^{a}_{n} E^{a}_{n} + B^{a}_{n} B^{a}_{n}\right) = \sum_{i} T_{ii}, \quad (8)$$

$$T_{0j} = -\varepsilon_{jmn} E^{a}_{m} E^{a}_{n}, \quad (9)$$

$$T_{ij} = -E^{a}_{i} E^{a}_{j} - B^{a}_{i} B^{a}_{j} + \delta_{ij} \frac{1}{2}\left(E^{a}_{n} E^{a}_{n} + B^{a}_{n} B^{a}_{n}\right). \quad (10)$$

Obviously, $T_{\mu\nu} = 0$ for any field configuration with $B^{a}_{n} = \pm i E^{a}_{n}$.

3. A Self Dual Solution of the Pure SU(2) Yang-Mills Theory

The pure SU(2) gauge theory is defined by the Lagrangian density

$$L = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a}. \quad (11)$$

Assuming that the gauge fields are radial, we use the Wu-Yang ansatz [8]

$$W^{a}_{0} = \frac{\hat{r}_a}{gr} H(r),\; W^{a}_{i} = \varepsilon_{aij} \frac{\hat{r}_j}{gr}\left[1 - I(r)\right], \quad (12)$$

where $H(r), I(r)$ are functions of *r*; $\hat{r}_a, \hat{r}_j$ are unit radial vector. Inserting this ansatz into the equation (5) yields two coupled nonlinear equations

$$r^2 H'' = 2HI,$$
$$r^2 I'' = I\left(I^2 - H^2 - 1\right), \quad (13)$$

where H'' and I'' denote differentiation with respect to *r*. The equations (13) have an explicit solution

$$I(r) = \frac{\alpha r}{\sinh \alpha r},$$
$$H(r) = \pm i\left(\alpha r \coth \alpha r - 1\right), \quad (14)$$

where α is an arbitrary constant and *i* is imaginary unit. The solution (14) is imaginary. This solution is different to the Prasad-Sommerfield solution [9], which is real.

The intensities of the pure SU(2) gauge fields are given by

$$E^{a}_{i} = F^{a}_{0i} = \frac{1}{g}\left[\left(\frac{H}{r}\right)' \hat{r}_i \hat{r}_a + \frac{HI}{r^2}\left(\delta_{ia} - \hat{r}_i \hat{r}_a\right)\right], \quad (15)$$

$$B^{a}_{i} = -\frac{1}{2}\varepsilon_{inm} F^{a}_{nm} = \frac{1}{g}\left[\left(\frac{1 - I^2}{r^2}\right)\hat{r}_i \hat{r}_a + \frac{I'}{r}\left(\delta_{ia} - \hat{r}_i \hat{r}_a\right)\right]. \quad (16)$$

Inserting (14) into (15) and (16), we find

$$E^{a}_{i} = \frac{\pm i}{g}\left[\left(\frac{1}{r^2} - \frac{\alpha^2}{\sinh^2 \alpha r}\right)\hat{r}_a \hat{r}_i + \left(\frac{\alpha^2 \coth \alpha \mathrm{r}}{\sinh \alpha r} - \frac{\alpha}{r \sinh \alpha r}\right)\left(\delta_{ai} - \hat{r}_a \hat{r}_i\right),\right] \quad (17)$$

$$B^{a}_{i} = \frac{1}{g}\left[\left(\frac{1}{r^2} - \frac{\alpha^2}{\sinh^2 \alpha r}\right)\hat{r}_a \hat{r}_i + \left(\frac{\alpha^2 \coth \alpha \mathrm{r}}{\sinh \alpha r} - \frac{\alpha}{r \sinh \alpha r}\right)\left(\delta_{ai} - \hat{r}_a \hat{r}_i\right),\right] \quad (18)$$

From the equations (17) and (18) we obtain $B^{a}_{i} = \pm i E^{a}_{i}$, therefore the solution (14) is self dual.

For $\alpha \neq 0$, the solution (14) corresponds to a broken local SU(2) symmetry because

$$I \to \alpha r e^{-\alpha r},\; H \to \pm i\alpha r,\; r \to \infty. \quad (19)$$

Thus α is the mass of the two Yang-Mills field components, which acquire a mass through the breaking of the local gauge symmetry.

4. A Self Dual Solution of the Pure SU(2) Yang-Mills Theory Connection with the Scalar ϕ^4 Theory

In Minkowski space the ansatz [10] for the SU(2) gauge potential connection with the scalar ϕ^4 theory is

$$W^{a}_{0} = \pm \frac{\partial_a \phi}{g\phi},$$

$$W^{a}_{i} = \pm i\delta_{ia} \frac{\partial_0 \phi}{g\phi} - \varepsilon_{ain} \frac{\partial_n \phi}{g\phi}, \quad (20)$$

where ϕ is a scalar function. The ansatz (20) is useful because it reduces the equation of motion (5) for the pure SU(2) gauge theory to a much simpler equation

$$\frac{1}{\phi}\partial_{\mu} \Box \phi = \frac{3}{\phi^2}\partial_{\mu} \phi \Box \phi, \quad (21)$$

where $\Box$ is D' Alembert operator. The equation (21) can be integrated once to give

$$\Box \phi + \lambda \phi^3 = 0. \quad (22)$$

In the equation (22), λ is an arbitrary integration constant. Suppose that solutions of the equation (22) are known, and in the ansatz (20) these lead automatically to explicit solutions of the SU(2) gauge theory. We see that ϕ can be interpreted as a physical field, and not merely as an ansatz function.

We now give formulas for several interesting quantities, following from the ansatz (20). The Yang-Mills electric field strength is

$$\begin{aligned} E_n^a &= \frac{1}{g}\varepsilon_{anm}\left[\frac{2}{\phi^2}\partial_0\phi\partial_m\phi - \frac{1}{\phi}\partial_0\partial_m\phi\right] \\ &\pm \frac{i}{g}\delta_{na}\left[\frac{1}{\phi}\partial_0^2\phi - \frac{1}{\phi^2}\left(\partial_0\phi\partial_0\phi + \partial_m\phi\partial_m\phi\right)\right] \\ &\mp \frac{i}{g}\left[\frac{1}{\phi}\partial_n\partial_a\phi - \frac{2}{\phi^2}\partial_n\phi\partial_a\phi\right]. \end{aligned} \quad (23)$$

The Yang-Mills magnetic field strength equals

$$B_n^a = \pm iE_n^a + \delta_{na}\left(\frac{1}{\phi}\right)\Box\phi. \quad (24)$$

From the equation (24) we see that the self dual condition $B_n^a = \pm iE_n^a$ evidently implies $\Box\phi = 0,$ or $\lambda = 0$ in the equation (22). The field strengths E_n^a and B_n^a are in general complex. However their squares are both real, and this means that the energy and Lagrangian density obtained from the ansatz (20) are real, even though the potential W_μ^a is complex.

The Lagrangian density (11) is given by

$$\begin{aligned} L &= \frac{1}{2g^2}\left[\Box\partial_\alpha\left(\frac{\partial^\alpha\phi}{\phi}\right) - \frac{1}{\phi}\Box\Box\phi\right] \\ &+ \frac{1}{2g^2\phi}\left[\frac{4}{\phi}\partial^\alpha\phi\partial_\alpha\Box\phi - \frac{6}{\phi^2}\Box\phi\partial^\alpha\phi\partial_\alpha\phi\right]. \end{aligned} \quad (25)$$

The energy-momentum tensor has the form

$$\begin{aligned} T_{\mu\nu} &= \frac{\Box\phi}{g^2\phi}\left[\frac{4}{\phi^2}\partial_\mu\phi\partial_\nu\phi - \frac{2}{\phi}\partial_\mu\partial_\nu\phi\right] \\ &+ \frac{\Box\phi}{g^2\phi}g_{\mu\nu}\left[\frac{1}{2\phi}\Box\phi - \frac{1}{\phi^2}\partial^\alpha\phi\partial_\alpha\phi\right]. \end{aligned} \quad (26)$$

If $\Box\phi = 0$ then equations (24) and (26) reduce to $B_n^a = \pm iE_n^a,\ T_{\mu\nu} = 0$ and the field is self dual.

In the case when scalar function ϕ is time independent and $\lambda = 0,$ the equation (22) has the spherically symmetric particular solution

$$\phi = \frac{c}{r}, \quad (27)$$

where c is an arbitrary constant. Inserting (27) into the ansatz (20) we find

$$W_0^a = \pm i\frac{\hat{r}_a}{gr},\quad W_i^a = \varepsilon_{ain}\frac{\hat{r}_n}{gr}, \quad (28)$$

where $\hat{r}_a, \hat{r}_n$ are unit radial vector. The solution (27) is self dual because which satisfies equation $\Box\phi = 0.$ We see that the spatial component $\left(W_i^a\right)$ of the SU(2) gauge potential in the equation (28) is analogus to the potential of a point magnetic monopole with magnetic charge.

5. Discussions and Conclusions

From the solution (14), if $\alpha \to 0$ then $I(r) \to 1,$ $H(r) \to 0,$ the SU(2) gauge potential becomes vacuum $\left(W_\mu^a = 0\right).$ For $\alpha \neq 0,$ the solution (14) is regular at $r = 0$ (because $I = 1$ and $H = 0$). This solution has topological charge $n = 1,$ as we see from the boundary condition $W_0^a = \hat{r}^a\frac{i\alpha}{g}$ at $r = \infty.$

To find explicit relation between the gauge potential W_μ and the scalar field ϕ we introduce Higgs triplet, which is defined by

$$\phi^a = \hat{r}^a\phi. \quad (29)$$

If scalar field ϕ is given by (27) then

$$\phi^a = \hat{r}^a\frac{c}{r}. \quad (30)$$

Because c is an arbitrary constant, one can chooses $c = \frac{1}{g}$ and the equation (30) is rewritten

$$\phi^a = \frac{\hat{r}^a}{gr}. \quad (31)$$

The equations (28) and (31) show that the gauge potential W_0^a can be reinterpreted as an imaginary Higgs field $i\phi^a$ or conversely ϕ^a as an imaginary gauge potential iW_0^a. Obviously, this is only true for static fields. In the general case, we consider an arbitrary SU(2) gauge transformation ω

$$\omega(x) = \exp\left(\frac{1}{2}i\sigma_a\theta_a(x)\right), \quad (32)$$

where σ_a are Pauli matrix. Under this transformation, the Higgs and gauge fields transform like

$$\begin{aligned} &\phi \to \omega\phi\omega^{-1}, \\ &W_\mu \to \omega W_\mu\omega^{-1} - (i/g)\left(\partial_\mu\omega\right)\omega^{-1}. \end{aligned} \quad (33)$$

If ω is time independent then $\partial_0\omega = 0$ and ϕ, W_0 transform in the same way. Therefore it is expected that W_0 and ϕ will contribute to gauge invariant quantities in much the same way.

In summary, by direct calculation we have given two explicit self dual solutions for the pure SU(2) Yang-Mills field and several interesting properties of the self dual fields. The solutions may be useful for some problems of non-Abelian gauge theories, for example, classical Higgs mechanism, magnetic monopole.

REFERENCES

[1] C. N. Yang, R. Mills, Conservation of isotopic spin and isotopic gauge invariance, *Phys. Rev. 96*, (1954), 191-195.

[2] J. Gattnar, K. Langfeld, H. Reinhardt, Signals of confinement in Green Functions of SU(2) Yang-Mills theory, *Phys. Rev. Lett. Vol. 93*, No. 6, (2004), 061601-1-4.

[3] Nguyen Vien Tho, To Ba Ha, Dynamical of global vortex string, *Jour. Math. Phys. 52*, (2011), 102302-1-9

[4] E. Radu, D. H. Tchrakian, Static BPS monopole in all even space-time dimension, *Phys. Rev. D 71*, (2005), 12503-1-8.

[5] M. Nielsen, N. K. Nielson, Explicit construction of constrained instantons, *Phys. Rev. D, Vol. 61*, (2000), 105020-1-15.

[6] A. S. Cornell, G. C. Joshi, Non-Abelian monopole and dyon solutions in a modified Einstein-Yang-Mills-Higgs system, *Phys. Rev. D 67*, (2003), 105015-1-11.

[7] D. Metaxas, Quantum-classical interactions through the path integral, *Phys. Rev. D 75*, (2007), 065023-1-7

[8] T. T. Wu and C. N. Yang, *Properties of matter under unusual conditions*, edited by H. Marl and S. Fernbach, Interscience, New York, (1969).

[9] M. K. Prasad and C. M. Sommerfield, Exact classical solution for the ʻt Hooft monopole and Julia-Zee dyon, *Phys. Rev. Lett. 35 (1975)* 760-762.

[10] F. Wilczek, *Quark confinement and field theory*, edited by D. Stump and D. Weingarten, Wiley, New York), (1977).

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