

CREATING NEW PROBLEMS ON PROVING INEQUALITIES, FINDING MAXIMUM AND MINIMUM VALUES BASED ON THE CRITICAL PROPERTIES AND TANGENT INEQUALITIES OF CONVEX AND CONCAVE FUNCTIONS

Huynh Duc Vu¹, Pham Quy Muoi^{2*}

¹Pham Van Dong High School, Quang Ngai

²The University of Danang - University of Science and Education

*Corresponding author: pqmuoi@ued.edu.vn

(Received May 25, 2021; Accepted June 20, 2021)

Abstract - In this paper, we present some ideas and methods to create new problems of proving inequalities, problems of finding maximum and minimum values. Based on the maximum and minimum properties and tangent inequalities of convex and concave functions, we propose some ideas and methods to create new problems. We make all ideas and methods to be real via many specific functions. Especially, we combine the ideas and methods with equivalent transforms, Cauchy-Schwarz inequality, and inequality of arithmetic and geometric means to create new hard problems. New proposed examples, they have showed that our ideas and methods are important and efficient to lecturers at high schools and universities in giving questions in examinations, especially in examinations of selecting good students at levels, in Olympic examinations for high school and university students.

Key words - Creating new problems; convex functions; Concave functions; tangent inequality; inequality; Maximum value; Minimum value.

1. Introduction

Convex calculus is a branch of mathematics devoted to the study of properties of convex sets, convex functions and related problems, which has many applications in optimization theory, control theory, partial differential equation theory,... and especially in proving important, fundamental inequalities. The theory of convex analysis has been studied and published in many different scientific works, typical of which can be mentioned the research works of [1, 4, 7, 8]. In high school math programs, the theory of convex and concave functions is also used quite commonly in proving problems about inequalities, to find the maximum and minimum values [3, 5, 6]. For example, the Cauchy - Schwarz inequalities, the Hölder inequality,.. are simply proved by applying the inequality necessary and sufficient conditions of the convex function [2]. However, the use of the theory of convex and concave functions to create new problems of proving inequalities, finding the maximum or minimum values of an expression is rarely mentioned. To our knowledge, this is a new direction, which has not been exploited and studied much.

In this paper, we introduce and propose some innovative methods to create new problems based on the basic properties of convex and concave functions. The basic idea of the proposed methods is a combination of the following three factors:

1. Using the properties of the extremes and the tangent inequalities of convex and concave functions;

2. Considering specific cases of convex and concave functions corresponding to different domains;

3. Combining methods of generalization, specialisation, equivalent transformations and common inequalities such as the Cauchy-Schwarz inequality, the AM-GM inequality, etc.

We will present some new problem creation ideas in detail based on the combination of the above three factors in the next part of this paper.

2. Creating new problems based on the extreme properties of convex and concave functions

In this section, we present ideas and methods to create new problems based on the extreme properties of convex and concave functions. Specifically, we rely on the following property:

Lemma 2.1 Let $f(x)$ be a function defined on $[x_1; x_2]$.

a) If $f(x)$ is a convex function on $[x_1; x_2]$ then

$$f(x) \leq \max\{f(x_1), f(x_2)\}, \forall x \in [x_1; x_2].$$

b) If $f(x)$ is a concave function on $[x_1; x_2]$ then

$$f(x) \geq \min\{f(x_1), f(x_2)\}, \forall x \in [x_1; x_2].$$

Proof. We will prove proposition a). Proposition b) will be proven similarly. With $x \in [x_1; x_2]$, there exists $\alpha \in [0,1]$ such that $x = \alpha x_1 + (1-\alpha)x_2$. Since f is a convex function, we have

$$\begin{aligned} f(x) &= f(\alpha x_1 + (1-\alpha)x_2) \\ &\leq \alpha f(x_1) + (1-\alpha)f(x_2) \\ &\leq \max\{f(x_1), f(x_2)\}. \end{aligned}$$

Thus, proposition a) is proven.

From this property we see that if we choose a specific convex function (concave function) $f(x)$, a particular domain $[x_1; x_2]$, we will get the value of $\max\{f(x_1), f(x_2)\}$ ($\min\{f(x_1), f(x_2)\}$). Then, we can create a problem of proving the inequality or problem of finding the maximum value (the problem of finding the minimum value). This is the main idea for creating inequalities based on this property. Note that, to create more difficult and diverse problems, we should combine with generalization or specializing methods, methods of changing variables. We will illustrate these ideas through two basic classes of functions: first-order functions and quadratic functions.

2.1. Creating new problems based on first order functions

First of all, we consider $f(x) = bx + c$ on $[0, a]$. Since function $f(x)$ is both convex and concave, $\forall x \in [0; a]$ we have

$$\min\{f(0), f(a)\} \leq f(x) \leq \max\{f(0), f(a)\}.$$

Furthermore, if we choose a, b, c such that $\max\{f(0), f(a)\} \leq 0$, then we can create the following inequality: Prove that

$$bx + c \leq 0, \forall x \in [0, a].$$

To increase the difficulty of the problem, we can add some parameters so that the values $f(0)$ and $f(a)$ depending on the parameters. For example, if we choose

$$f(a) = -yz \leq 0,$$

$$f(0) = -(a - y)(a - z) \leq 0, \forall y, z \in [0; a],$$

then we have

$$f(x) \leq \max\{f(0), f(a)\} \leq 0.$$

Using the conditions on $f(0)$ and $f(a)$ we get b, c and function $f(x) = (a - y - z)x + a(y + z) - yz - a^2$ on $[0, a]$. Then, the inequality $f(x) \leq 0$ for all $x \in [0, a]$ is equivalent to $a(x + y + z) - (xy + yz + zx) \leq a^2, \forall x, y, z \in [0; a]$. Thus, we have the following problem:

Problem 2.2 Let a be a fixed positive real number and x, y, z in $[0; a]$. Prove that

$$a(x + y + z) - (xy + yz + zx) \leq a^2. \quad (2.1)$$

From inequality (2.1), we can create a number of different inequality problems for each parameter value. For example, we have the following new problems by giving different values for the parameter a . For example, we have the following problems by letting $a = 3$ and $a = 2020$.

Exercise 2.3 Let $x, y, z \in [0; 3]$. Prove that

$$3(x + y + z) - (xy + yz + zx) \leq 9.$$

Exercise 2.4 Let $x, y, z \in [0; 2020]$. Find the maximum value of the expression

$$P = 2020(x + y + z) - (xy + yz + zx).$$

To create new problems with increasing difficulty, we can choose $f(0)$ and $f(a)$ depending on some parameters so that the value of $\max\{f(0), f(a)\}$ depends on those parameters. Furthermore, we can use methods of changing variables and add additional conditions on variables to obtain new problems of proving inequalities or problems of finding the maximum and minimum values for a multivariable expression in which the variables change depending on each other through some constraint conditions.

To illustrate the above idea, we consider the following specific examples. We still start from the first order function $f(t) = bt + c$ with $t \geq 0$ and $f(0) = x(1 - x), x \in [0, 1]$. Then, $f(t) = bt + x(1 - x)$. To create a symmetric three-variable inequality, we can choose $b = 1 - ax$, let $t = yz$ with the condition $y, z \geq 0$ and $x + y + z = 1$. Substituting these values into the expression $f(t)$ we get

$$P = xy + yz + zx - axyz$$

$$= f(yz) = (1 - ax)yz + x(1 - x).$$

From the conditions $y, z \geq 0$ and $x + y + z = 1$ we obtain

$$0 \leq yz \leq \left(\frac{y+z}{2}\right)^2 = \left(\frac{1-x}{2}\right)^2.$$

Thus, we get the function

$$f(yz) = (1 - ax)yz + x(1 - x), yz \in \left[0; \left(\frac{1-x}{2}\right)^2\right]$$

and $P = xy + yz + zx - axyz$

$$= f(yz) \leq \max\left\{f(0), f\left(\left(\frac{1-x}{2}\right)^2\right)\right\}, x \in [0; 1].$$

From the above results, we have the following new problem:

Problem 2.5 Let $x, y, z \geq 0, x + y + z = 1$. For each $a \neq 0$, find the maximum value of the expression

$$P = xy + yz + zx - axyz.$$

Similar to the previous example, if we choose specific values for a , we have new problems of finding the maximum value of $T = xy + yz + zx - axyz$. For example, we can state some new problems:

Exercise 2.6 Let $x, y, z \geq 0, x + y + z = 1$. Find the maximum value of the expression $T = xy + yz + zx - 3xyz$.

Exercise 2.7 Let $x, y, z \geq 0, x + y + z = 1$. Find the maximum value of the expression $T = 2(xy + yz + zx) - xyz$.

Exercise 2.8 Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

2.2. Creating new problems based on quadratic functions

Now, let us move on to creating new problems through quadratic functions $f(x) = ax^2 + bx + c$ on $[x_1; x_2]$. We know that if $a > 0$ then $f(x)$ is convex on $[x_1; x_2]$. Otherwise, if $a < 0$ then $f(x)$ is concave on $[x_1; x_2]$. Thus, if we choose a particular convex (concave) quadratic function and a particular interval $[x_1; x_2]$, then we obtain a particular problem of proving the inequality or a particular problem of finding the maximum value (finding the minimum value), respectively. These problems, although new, but are quite simple. To create new and more difficult problems, we can use the same ideas as presented for the first-order functions. Here, we will present another method to create new problems in the case f is a quadratic function.

The basic idea of this direction is based on the following result: if function $P(x, y)$ satisfies "for each fixed $x, P(x, \cdot)$ is a convex quadratic function with respect to y and for each fixed $y, P(\cdot, y)$ is a convex quadratic function with respect to x " then

$$\begin{aligned} P(x, y) &\leq \max\{P(a, y), P(b, y)\} \\ &\leq \max\{P(a, a), P(a, b), P(b, a), P(b, b)\}. \end{aligned}$$

Thus, we can create problems of proving inequalities or problems of finding the maximum values with respect to each choice of function P and each interval $[a, b]$.

Similarly, if the function $P(x, y)$ satisfies: "for each fixed x , $P(x, \cdot)$ is a concave quadratic function with respect to y and for each fixed y $P(\cdot, y)$ is a concave quadratic function with respect to x ." We can create problems of proving inequalities or problems of finding the minimum value for each choice of function P and interval $[a, b]$. We will illustrate this idea through the following specific examples.

Consider the function

$$f(x) = (x + y + z)(yz + 2xz + 3xy) - \frac{80}{3}xyz.$$

This is a quadratic function with coefficient $a = 2z + 3y > 0$ (we assume that x, y, z in $[1; 3]$). $f(x)$ is convex $[1; 3]$. Therefore, $f(x) \leq \max\{f(1), f(3)\}$. Note that

$$\bullet f(1) = (1 + y + z)(yz + 2z + 3y) - \frac{80}{3}yz = g_1(y),$$

$$g_1(1) = \frac{1}{3}(9z^2 - 53z + 18), g_1(3) = 5z^2 - 51z + 36.$$

$$\bullet f(3) = (3 + y + z)(yz + 6z + 9y) - 80yz = g_3(y),$$

$$g_3(1) = 7z^2 - 43z + 36, g_3(3) = 9z^2 - 159z + 162.$$

Since $g_1(y), g_3(y)$ are also convex on $[1; 3]$, for any $y \in [1; 3]$ we have

$$g_1(y) \leq \max_{z \in [1,3]} \{g_1(1), g_1(3)\} = -\frac{26}{3},$$

$$g_3(y) \leq \max_{z \in [1,3]} \{g_3(1), g_3(3)\} = 0.$$

Therefore, $\forall x, y, z \in [1, 3]$ we have

$$(x + y + z)(yz + 2xz + 3xy) - \frac{80}{3}xyz \leq 0.$$

To make the problem more difficult, we can divide both sides by xyz , then reduce and shift some terms from the left side to the right side. For example, we can state the problem as follows:

Problem 2.9 Let $x, y, z \in [1; 3]$. Prove that

$$(x + y + z) \left(\frac{1}{x} + \frac{2}{y} + \frac{3}{z} \right) \leq \frac{80}{3}.$$

Similar to the above, we can create the following new problems:

Exercise 2.10 Let $x, y, z \in [1; 2]$. For each given set of three positive numbers a, b, c , find the maximum value of the following expression in terms of a, b, c :

$$P = (x + y + z) \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \right).$$

Exercise 2.11 Let $x, y, z \in [1; 3]$. For each given set of three positive numbers a, b, c , find the maximum value of the following expression in terms of a, b, c :

$$Q = (x + y + z) \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \right).$$

Exercise 2.12 Let $x, y, z \in [1; 3]$. Prove that

$$(x + y + z) \left(\frac{2020}{x} + \frac{21}{y} + \frac{12}{z} \right) \leq 14217.$$

3. Creating new problems based on tangent inequalities of convex and concave functions

In this section, we present ideas and methods for creating new problems based on the continuation inequality of convex and concave functions. Specifically, we rely on the following properties of convex and concave functions:

Lemma 3.1 [2,p.176] Let $f(x)$ be a differentiable function on $[x_1; x_2]$.

• If $f(x)$ is convex on $[x_1; x_2]$ then for each $x_0 \in [x_1; x_2]$ we have

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0), \forall x \in [x_1; x_2].$$

• If $f(x)$ is concave on $[x_1; x_2]$ then for each $x_0 \in [x_1; x_2]$ we have

$$f(x) \leq f(x_0) + f'(x_0)(x - x_0), \forall x \in [x_1; x_2].$$

The point $x_0 \in [x_1; x_2]$ in the above property is called the "falling point" and the two inequalities in Lemma 3.1 are called the *tangent inequalities for convex and concave functions*, respectively. Thus, if $f(x)$ is a differentiable convex function on $[x_1; x_2]$ and x_0 a falling point, then for all real numbers $a_1, a_2, \dots, a_n \in [x_1; x_2]$ we have

$$f(a_1) \geq f(x_0) + f'(x_0)(a_1 - x_0),$$

$$f(a_2) \geq f(x_0) + f'(x_0)(a_2 - x_0),$$

...

$$f(a_n) \geq f(x_0) + f'(x_0)(a_n - x_0).$$

Adding n inequalities on both sides we get

$$\sum_{i=1}^n f(a_i) \geq nf(x_0) + f'(x_0)(\sum_{i=1}^n a_i - nx_0). \quad (3.1)$$

If $f(x)$ is strictly convex, then the equal sign in the above inequality occurs if and only if $a_i = x_0$ for all $i = 1, \dots, n$. From (3.1), we see that if we choose a particular convex function f , a particular falling point and a set of numbers $a_1, a_2, \dots, a_n \in [x_1; x_2]$, then we can get a problem of proving the inequality. Furthermore, if we set the condition $\sum_{i=1}^n a_i = S$ (constant), we can get a problem of finding the minimum value of the expression on the left side. In the case of a differentiable and concave function $f(x)$ on $[x_1; x_2]$, repeating the above process, we also get problems of proving inequalities or the problems of finding the maximum value of the expression on the left side. We will illustrate how to create such new problems through the following specific examples.

Considering the function $f(x) = \frac{x}{x^2+1}$ on $[0; 1]$, we have

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^2} \leq 0, \forall x \in [0; 1].$$

Function $f(x)$ is concave on $[0; 1]$. With falling point $x_0 = \frac{1}{3}$ and for any $a, b, c \in [0; 1]$ we have

$$f(a) \leq f'\left(\frac{1}{3}\right) \left(a - \frac{1}{3}\right) + f\left(\frac{1}{3}\right),$$

$$f(b) \leq f'\left(\frac{1}{3}\right) \left(b - \frac{1}{3}\right) + f\left(\frac{1}{3}\right),$$

$$f(c) \leq f'\left(\frac{1}{3}\right) \left(c - \frac{1}{3}\right) + f\left(\frac{1}{3}\right).$$

Adding up the above inequality, we get

$$P = f(a) + f(b) + f(c)$$

$$\leq f'\left(\frac{1}{3}\right) (a + b + c - 1) + 3f\left(\frac{1}{3}\right).$$

If we set the condition $a + b + c = 1$, then P reaches the maximum value if and only if $a = b = c = \frac{1}{3}$. So we have the following problem:

Problem 3.2 Let a, b, c be nonnegative real numbers such that $a + b + c = 1$. Find the maximum value of the

expression

$$P = \frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1}.$$

Similar to the above, we can also come up with a new problem of proving the inequality:

Problem 3.3 Let x_1, x_2, \dots, x_n be nonnegative real numbers such that $x_1 + x_2 + \dots + x_n = na$ with $a > 0$. Prove that

$$\frac{x_1}{x_1^2+1} + \frac{x_2}{x_2^2+1} + \dots + \frac{x_n}{x_n^2+1} \leq \frac{na}{a^2+1}.$$

The equality occurs if and only if $x_1 = x_2 = \dots = x_n = a$.

Similarly, if we choose the concave function $f(x) = \frac{x}{\sqrt{x^2+12}}$, we can create many problems of finding maximum value or problems of proving inequalities as follows:

Problem 3.4 Let a, b, c be positive real numbers such that $a + b + c = 6$. Find the minimum value of the expression

$$P = \frac{a}{\sqrt{a^2+12}} + \frac{b}{\sqrt{b^2+12}} + \frac{c}{\sqrt{c^2+12}}$$

Exercise 3.5 Let x_1, x_2, \dots, x_n be positive real numbers such that $x_1 + x_2 + \dots + x_n = na$ with $a > 0$. Find the maximum value of the expression

$$P = \sqrt{\frac{x_1}{x_1^2+1}} + \sqrt{\frac{x_2}{x_2^2+1}} + \dots + \sqrt{\frac{x_n}{x_n^2+1}}$$

To make problems more difficult, we can combine inequality (3.1) with some other inequalities such as the Cauchy-Schwarz inequality. We illustrate this through the following specific example:

Consider the function $f(x) = \frac{1}{\sqrt{2020+3x}}$ with $0 < x \leq \sqrt{3}$. We have

$$f'(x) = -\frac{3}{2\sqrt{(2020+3x)^3}}$$

$$f''(x) = \frac{27}{4\sqrt{(2020+3x)^5}} > 0, \forall x \in (0; \sqrt{3}).$$

Then, $f(x)$ is convex on $(0; \sqrt{3})$. Let $x_0 = 1$ be a falling point. Using the tangent inequality, for any $a, b, c \in (0; \sqrt{3})$ we have

$$f(a) \geq f'(1)(a - 1) + f(1),$$

$$f(b) \geq f'(1)(b - 1) + f(1),$$

$$f(c) \geq f'(1)(c - 1) + f(1).$$

Now if we add the condition $a^2 + b^2 + c^2 = 3$ and use Cauchy-Schwarz inequality, then we have

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2) = 9.$$

$$\text{It is implied that } a + b + c - 3 \leq 0.$$

Adding the above tangent inequalities, we get

$$\begin{aligned} & \frac{1}{\sqrt{2020+3a}} + \frac{1}{\sqrt{2020+3b}} + \frac{1}{\sqrt{2020+3c}} \\ & \geq f'(1)(a + b + c - 3) + 3f(1) \\ & \geq 3f(1) = \frac{3}{\sqrt{2021}} \end{aligned}$$

$$\text{Since } f'(1) = -\frac{4}{27} < 0.$$

Thus, we can pose the following problem:

Problem 3.6 Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Find the minimum value of the expression

$$P = \frac{1}{\sqrt{2020+3a}} + \frac{1}{\sqrt{2020+3b}} + \frac{1}{\sqrt{2020+3c}}$$

To make the problem more difficult to identify and select the function, we should combine the tangent inequality with the equivalence transformations or use it in combination with other inequalities such as the Cauchy-Schwarz inequality, inequality of arithmetic and geometric means (AM-GM inequality). The following two examples illustrate and further clarify this combination. The first example is the combination of the triangle inequality and the equivalence transformation. The second example illustrates a combination of the equivalence transformation, Cauchy-Schwarz inequality, AM-GM inequality, and the tangent inequality.

Consider the function $f(t) = \ln t$. We have $f''(t) = -\frac{1}{t^2} < 0, \forall t > 0$. Choosing the falling point $t = \frac{1}{3}$ and using the tangent inequality we have for any $x > 0$

$$\ln x \leq f'\left(\frac{1}{3}\right)\left(x - \frac{1}{3}\right) + f\left(\frac{1}{3}\right) = 3x - 1 - \ln 3.$$

Multiplying both sides of the above inequality with $y > 0$ we have

$$y \ln x \leq 3xy - y - y \ln 3.$$

Similarly, we also have the following inequalities: for all $x, y, z > 0$,

$$z \ln y \leq 3yz - z - z \ln 3,$$

$$x \ln z \leq 3zx - x - x \ln 3.$$

Adding the last three inequalities on both sides and applying the Cauchy-Schwarz's inequality, we get:

$$\begin{aligned} \ln A &= y \ln x + z \ln y + x \ln z \\ &\leq 3(xy + yz + zx) - 1 - \ln 3 \\ &\leq (x + y + z)^2 - 1 - \ln 3. \end{aligned}$$

If we add the condition $x + y + z = 1$ then we have $\ln A \leq -\ln 3$ or $A = x^y y^z z^x \leq \frac{1}{3}$. On the other hand, using AM-GM inequality, we have

$$P = \frac{1}{x^y} + \frac{1}{y^z} + \frac{1}{z^x} \geq \frac{3}{\sqrt[3]{x^y y^z z^x}} \geq 3\sqrt[3]{3}.$$

Thus, P reaches the minimum value that is $3\sqrt[3]{3}$ as $x = y = z = \frac{1}{3}$. Based on this analysis, we can create two problems with different difficulty as follows:

Exercise 3.7 Let $x, y, z > 0$ such that $x + y + z = 1$. Find the maximum value of the expression

$$P = x^y y^z z^x.$$

Exercise 3.8 Let $x, y, z > 0$ such that $x + y + z = 1$. Find the minimum value of the expression

$$P = \frac{1}{x^y} + \frac{1}{y^z} + \frac{1}{z^x}.$$

Consider the function $f(x) = x^{2021}$ with $x > 0$. We have $f'(x) = 2021x^{2020}, f''(x) > 0, \forall x > 0$.

Select the falling point $x_0 = \frac{1}{1011}$. Then, for any $x_i >$

$0, i = 1, 2, \dots, 2021$, using the tangent inequality we have

$$\begin{aligned} f(x_i) &\geq f'\left(\frac{1}{1011}\right)\left(x_i - \frac{1}{1011}\right) + f\left(\frac{1}{1011}\right) \\ &= \frac{2021}{1011^{2020}}x_i - \frac{2020}{1011^{2021}}. \end{aligned}$$

This inequality is equivalent to (multiplying both sides with $i > 0$)

$$ix_i^{2021} \geq \frac{2021}{1011^{2020}}ix_i - \frac{2020i}{1011^{2021}}, \forall i = 1, 2, \dots, 2021.$$

Adding all these inequalities on both sides, we get

$$\begin{aligned} \sum_{k=1}^{2021} kx_k^{2021} &\geq \frac{2021}{1011^{2020}} \cdot (x_1 + 2x_2 + \dots + 2021x_{2021}) \\ &\quad - \frac{2020}{1011^{2021}} \cdot (1 + 2 + \dots + 2021) \\ &\geq \frac{2021}{1011^{2020}} \cdot 2021 - \frac{2020}{1011^{2020}} = \frac{4282500}{1011^{2020}}. \end{aligned}$$

The equality occurs if and only if $x_1 = x_2 = \dots = x_n = \frac{1}{1011}$. Thus, we can have a new problem as follows:

Exercise 3.9 Let $x_1, x_2, \dots, x_{2021}$ be positive real numbers such that $\sum_{k=1}^{2021} kx_k = 2021$. Find the minimum value of the expression

$$P = \sum_{k=1}^{2021} kx_k^{2021}.$$

To conclude the presentation of ideas and methods for creating problems based on tangent inequalities, we consider the choice of functions (convex or concave) depending on one or more parameters. The problems created in this case are quite complex and often difficult to solve. As an illustrative example, we consider a function $f(a) = a^3 + (6b + 9)a^2$ with $a > -1$, b is a parameter and $b > -1$. We have

$$\begin{aligned} f'(a) &= 3a^2 + 2(6b + 9)a, \\ f''(a) &= 6(a + 2b + 3) > 0, \forall a, b > -1. \end{aligned}$$

Let us select the falling point $x_0 = 1$. Then, for any $a \in (-1; +\infty)$,

$$\begin{aligned} f(a) &\geq f'(1)(a - 1) + f(1), \\ \Leftrightarrow a^3 + (6b + 9)a^2 &\geq (12b + 21)(a - 1) + (6b + 10). \end{aligned}$$

Similarly, we get for any $b, c \in (-1; +\infty)$,

$$\begin{aligned} b^3 + (6c + 9)b^2 &\geq (12c + 21)(b - 1) + (6c + 10), \\ c^3 + (6a + 9)c^2 &\geq (12a + 21)(c - 1) + (6a + 10). \end{aligned}$$

Adding the last three inequalities, we have

$$\begin{aligned} a^3 + b^3 + c^3 + 6(a^2b + b^2c + c^2a) + 9(a^2 + b^2 + c^2) \\ \geq 12(ab + bc + ca) + 15(a + b + c) - 33. \end{aligned}$$

Thus, if we add the condition such that the right hand side is constant number then we can create a new problem as follows:

Problem 3.10 Let $a, b, c > -1$ and $4(ab + bc + ca) + 5(a + b + c) \geq 27$. Find the minimum value of the expression

$$\begin{aligned} P &= a^3 + b^3 + c^3 + 6(a^2b + b^2c + c^2a) \\ &\quad + 9(a^2 + b^2 + c^2). \end{aligned}$$

We see that the condition in the above problem is quite complex, we can replace it with a simpler condition by using the Cauchy-Schwarz inequality. For example we

can replace the condition $4(ab + bc + ca) + 5(a + b + c) \geq 27$ by $ab + bc + ca = 3$. Then, applying the Cauchy-Schwarz inequality, we have

$$a + b + c \geq \sqrt{3(ab + bc + ca)} = 3.$$

From this inequality and the newly introduced condition, we get the condition:

$$4(ab + bc + ca) + 5(a + b + c) \geq 27.$$

Using the above approach, that is, combining the tangent inequality and the Cauchy-Schwarz inequality, for the function $f(a) = 3a^{10} + 5ba^3$ where $a > 0$, b is a parameter, $b > 0$ and the falling point is $x_0 = 1$, we have the following new problem:

Problem 3.11 Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Find the minimum value of the expression

$$P = 3(a^{10} + b^{10} + c^{10}) + 5(a^3b + b^3c + c^3a).$$

From the above examples, readers can predict and choose suitable functions to create the following new exercises.

Exercise 3.12 Let $a, b, c \in [1; \sqrt{6}]$ and $ab + bc + ca + 24 = 6(a + b + c)$. Find the maximum value of the expression

$$P = \frac{b}{a} + \frac{a}{c} + \frac{c}{b} - 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

Exercise 3.13 Let $a, b, c \in [0; 3]$ such that $ab + bc + ca \geq 3$. Find the maximum value of the expression

$$b\sqrt{3a+1} + c\sqrt{3b+1} + a\sqrt{3c+1} - (a^2 + b^2 + c^2)$$

4. Conclusion

The main result of this paper is to present methods of creating new problems of proving inequalities and finding the maximum (or minimum) values of convex (or concave) functions. We illustrate the methods via some specific choice of functions. Through these examples, we see that the methods are very easy to use, and we can create many new problems with different difficulties. Therefore, the methods are very useful for teachers and lecturers to create new exercises and problems for exams.

REFERENCES

- [1] Jonathan Borwein and Adrian S Lewis. *Convex analysis and nonlinear optimization: theory and examples*. Springer Science & Business Media, 2010.
- [2] Jean-Marie Monier (Lí Hoàng Tú Dịch). *Giáo trình Toán Tập 1: Giải tích I*. Nhà xuất bản Giáo dục Việt Nam, 2013.
- [3] Nguyễn Quý Duy. *Tuyển tập 200 bài vô địch toán*. Nhà xuất bản Giáo dục Hà Nội, 2001.
- [4] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Fundamentals of convex analysis*. Springer Science & Business Media, 2004.
- [5] Đỗ Văn Lưu. *Giải tích hàm*. NXB Khoa học và Kỹ thuật, Hà Nội, 1999.
- [6] Nguyễn Văn Mậu. *Bát đẳng thức, Định lý và áp dụng*. Nhà xuất bản Giáo dục, 2005.
- [7] R Tyrrell Rockafellar. *Convex analysis*, volume 36. Princeton university press, 1970.
- [8] Josef Stoer and Christoph Witzgall. *Convexity and optimization in finite dimensions I*, volume 163. Springer Science & Business Media, 2012.