APPLYING SEMISMOOTH NEWTON METHOD TO FIND FIXED POINTS OF NONSMOOTH FUNCTIONS OF ONE VARIABLE

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Abstract - In this paper, we investigate the problem of finding a fixed point of the nonsmooth function, $\max\ f_1(x), f_2(x),\ldots, f_n(x)$. First, we recall the definition of Newton derivative and examine some basic properties. Then, we investigate the Newton differentiability of function $\max\ f_1(x), f_2(x),\ldots, f_n(x)$. We give the necessary and sufficient conditions for Newton differentiability of this function in two cases: A special case: $\max\ f_1(x), f_2(x)$ and the general case: $\max\ f_1(x), f_2(x),\ldots, f_n(x)$. We emphasize that, the sufficient condition for the special case is much weaker than that of the general case. After that, we apply the semismooth Newton method to find a fixed point of the above function. The local quadratic order convergence of the method is proven. Finally, we present the numerical results for some specific examples.

Key words - Newton Derivative; Newton differential; Fixed point; Semismooth Newton method; Nonsmooth function

1. Introduction

Thought out the history, fixed point theory has been widely considered by numerous researchers both domestically and internationally. There is a large amount of published research including Banach, Browder and Borel's fixed point theory, ... [1, 4]. Fixed point theory has wide applications in abundant areas such as partial differential equation theory, economics (game theory), ... [3, 4]. In numerical programming, the familiar method which has been used is fixed point's iteration as well as advanced one [2]. As we know, fixed point iteration converges in linear speed.

Recently, in optimization theory for non-smooth problems, there are more advanced algorithms with high convergent speed that have been studied and improved. In these algorithms, semismooth Newton's method for fixed point problems is widely researched and applied. In this paper, we investigate semismooth Newton's method for fixed point problems and provide quadratic convergence of our algorithm. Specially, we study this method for fixed point problems:

$$F(x) = \max\ f_1(x), f_2(x),\ldots, f_n(x)\ , \tag{1}$$

where $f_i : C \to \mathbb{R}$ with $\varnothing \neq C \subseteq \mathbb{R}$ ($i = 1,\ldots,n$) is smooth, continuous functions.

Problem (1) appears in different areas, especially in constrained optimization problems. Some special cases are most likely to find in the necessary condition of solution in sparsity problems [7] and non-sparsity problems [8].

Noticeably, fixed point problem of F is equivalent to finding solution to the following equation

$$G(x) := x - F(x) = 0. \tag{2}$$

We want to emphasize that function F is continuous function but non-smooth so that G is non-smooth function too. Hence, the efficient method to find solution to this function like gradient method and Newton method, ... that are not utilized. In this paper, we investigate Newton differentiable of F and apply semismooth Newton to find solution to equation (2). Finally, we apply this method in some examples.

The other parts of this paper are organized as follows. In section 2, we first re-define Newton derivative for one variable and recall some its properties. In section 3, we present Newton differentiable of F with $n = 2$. In section 4, we state Newton differentiable of F for $n \geq 2$ in general. In section 5, we introduce and prove the convergence of semismooth Newton method for equation (2). In conclusion, we also present many numerical examples.

2. Newton derivative

Definition 2.1. Let U be a open subset of $\mathbb{R}$, $F : U \to \mathbb{R}$ be a function that defines in U. Function F is Newton differentiable at $x \in U$ if exist function $G : U \to \mathcal{L}(U,\mathbb{R})$ so that

$$\lim_{h\to 0} \frac{|\ F(x+h) - F(x) - G(x+h)h\ |}{|\ h\ |} = 0.$$

where $\mathcal{L}(U,\mathbb{R})$ is the space of all linear bounded mappings from U to $\mathbb{R}$.

The function G is called one of Newton derivative F at x.

From the previous definition, we imply that Newton derivative of F at one point is a function and it is not a real number like Fréchet derivative.

Remark 2.2. *In [7], authors indicated that:*

a) Newton derivative is not singular.

b) If function F is continuous and Fréchet differentiable in (a,b) then F is Newton differentiable in (a,b). Moreover, F' is one of Newton derivative of function F.

c) If function f and g are Newton differentiable at x then $\lambda f, f \pm g, fg$ are Newton differentiable and one of their Newton derivative is equivalent to Fréchet derivative [5].

Next, we consider Newton derivative of function F that was given by (1). Because of the distinction of Newton differentiable of function F when $n = 2$ and general case ($n \geq 2$) we will present its differentiable in individual cases. In addition, the sufficient condition to reach Newton differentiable of F when $n=2$ (that is presented in [5]) is weaker than this one in general case.

3. Newton derivative of F when $n=2$

Newton derivative of F when $n=2$ was presented in [5]. For the convenience for readers, we recall some important properties in [5] by the following theorem.

Theorem 3.2. *Let f and g be continuously Fréchet differentiable in $\mathbb{R}$ then function $F(x)=\max\{f(x),g(x)\}$ is Newton differentiable for all $x\in\mathbb{R}$ and one of Newton derivative of $F(x)$ is G which is defined by*

$$G(x)=\begin{cases} f'(x),\ x\in P\\ g'(x),\ x\in Q,\end{cases}$$

where,

$P=\{x \mid f(x)>g(x)\}$ *and* $Q=\{x \mid f(x)\le g(x)\}$.

4. Newton derivative of F when $n\ge 2$

Theorem 4.3. *Let $f_1(x),\ldots,f_n(x)$ be n continuously differentiable function in $\mathbb{R}$ which has finite intersections.*

Denote $F(x)=\max\{f_1(x),\ldots,f_n(x)\}$,

$$I(x)=\{i\in\{1,2,\ldots,n\} \mid f_i(x)=F(x)\},$$

$$A(x)=\min I(x).$$

Then, $F(x)$ is Newton differentiable for all $x_0\in\mathbb{R}$ and one of Newton derivatives of $F(x)$ at x_0 is

$$G(x)=f'_{A(x)}(x), x\in\mathbb{R}.$$

Prove. For all $x_0\in\mathbb{R}$, we consider two cases:

Case 1: $I(x_0)=\{k\}, k\in\{1,2,..,n\}$.

Then $f_k(x_0)\ge f_j(x_0), \forall j\in\{1,..,n\}, j\ne k$.

Since f_i is continuous in $\mathbb{R}$ for all $i\in 1,..,n$ then for all $x\in\mathbb{R}$, there must be a $\delta>0$ that satisfies $x\in(x_0-\delta,x_0+\delta)$ and we have

$$f_k(x)\ge f_j(x), \forall j\in\{1,..,n\}, j\ne k.$$

For h so that $x_0+h\in(x_0-\delta,x_0+\delta)$, we have

$$0\le\frac{|F(x_0+h)-F(x_0)-G(x_0+h)h|}{|h|}$$
$$=\frac{|f_k(x_0+h)-f_k(x_0)-f'_k(x_0+h)h|}{|h|}$$
$$\le\frac{|f_k(x_0+h)-f_k(x_0)-f'_k(x_0)h|}{|h|}+|f'_k(x_0)-f'_k(x_0+h)|$$
$$\to 0, h\to 0.$$

The last one follows from the continuity and differentiability of function $f_k(x)$ in $\mathbb{R}$.

Case 2: $I(x_0)$ contains more than one element.

Because of the finite intersections, there must be a $\delta>0$ so that $I(x)=\{k\},\forall x\in(x_0-\delta,x_0)$ and $I(x)=\{l\},\forall x\in(x_0,x_0+\delta)$.

+ For h so that $x_0+h\in(x_0-\delta,x_0)$ we have

$$0\le\frac{|F(x_0+h)-F(x_0)-G(x_0+h)h|}{|h|}$$
$$=\frac{|f_k(x_0+h)-f_k(x_0)-f'_k(x_0+h)h|}{|h|}$$
$$\le\frac{|f_k(x_0+h)-f_k(x_0)-f'_k(x_0)h|}{|h|}+|f'_k(x_0)-f'_k(x_0+h)|$$
$$\to 0, \text{as } h\to 0.$$

The last one follows from the continuity and differentiability of function $f_k(x)$ in $\mathbb{R}$.

+ For h so that $x_0+h\in(x_0,x_0+\delta)$ we have

$$0\le\frac{|F(x_0+h)-F(x_0)-G(x_0+h)h|}{|h|}$$
$$=\frac{|f_l(x_0+h)-f_l(x_0)-f'_l(x_0+h)h|}{|h|}$$
$$\le\frac{|f_l(x_0+h)-f_l(x_0)-f'_l(x_0)h|}{|h|}+|f'_l(x_0)-f'_l(x_0+h)|$$
$$\to 0, h\to 0.$$

The last one follows from the continuity and differentiability of function $f_l(x)$ in $\mathbb{R}$.

5. Semismooth Newton's method for finding fixed point

Consider function $F(x)=\max\{f_1(x),\ldots,f_n(x)\}$ where $f_i(x),\ i=\overline{1,n}$ is continuously Fréchet differentiable which has finite intersections.

Denote $G(x)=x-F(x)$ and consider the following problem

$$G(x)=0 \qquad (3)$$

Denote x^* is a solution to function (3) then the iteration of semismooth Newton method for problem (3) is given by

$$x_{n+1}=x_n-\frac{1}{G'(x_n)}G(x_n), \qquad (4)$$

where G' is one of Newton derivative of G.

Theorem 5.4. Assume x^* is one solution to problem

(3) and function G is Newton differentiable with G' the Newton derivative of G at x^*. If there is a neighborhood U of x^* so that $f_i'(x) \neq 1, \forall x \in U, i = \overline{1,n}$ then Newton iteration (4) converges to x^* when $|x_0 - x^*|$ is small enough.

Prove. We have $G(x^*) = 0$.

$$|x_{k+1} - x_*| = |x_k - \frac{1}{G'(x_k)} G(x_k) - x^*|$$

$$= \left| \frac{1}{G'(x_k)} [G'(x_k)(x_k - x^*) + G(x^*) - G(x_k))] \right|$$

$$\leq \left| \frac{1}{G'(x_k)} \right| \left| G(x_k) - G(x^*) - G'(x_k)(x_k - x^*) \right| \quad (5)$$

Since $f_i'(x) \neq 1, \forall x \in U(x^*), \forall i = \overline{1,n}$

it follows that $G'(x) = 1 - f'_{A(x)} \neq 0, \forall x \in U(x^*)$.

This implies that $\frac{1}{G'(x)}$ is bounded in U.

Choose $\delta > 0$ so that $(x^* - \delta, x^* + \delta)$ contains U and $M > 0$ so that

$$\left| \frac{1}{G'(x)} \right| \leq M, \forall x \in (x^* - \delta, x^* + \delta).$$

Since G' is Newton derivative of G at x^* then exists $r \in (0, \delta)$ so that

$$|G(x^* + h) - G(x^*) - G'(x^* + h)h| < \frac{|h|}{2M}, \quad (6)$$

for all $|h| < r$. In addition, by choosing x_0 so that $|x_0 - x^*| < r$ then from (5), (6) and $h = x_k - x^*$ we deduce that

$$x_k \in (x^* - \delta, x^* + \delta), \forall k \text{ and}$$

$$|x_{k+1} - x^*| < M.\frac{1}{2M}|h| = \frac{1}{2}|x_k - x^*|.$$

Therefore, iteration (4) is completely defined and $x_k \to x^*$.

Example 5.1. Find the fixed point of function $F(x) = \max\{2x^3 - x, x^2\}$.

Denote $G(x) = x - F(x)$.

According to the prove in previous theorems, G is Newton differentiable. Noticeably, $x^* = 0$ and $x^* = 1$ are zero-point of $G(x)$. Moreover, Newton derivative G' satisfies assumption in Theorem 5.1. Therefore, we apply semismooth Newton method that is given by

$$x_{n+1} = x_n - \frac{1}{G'(x_n)} G(x_n).$$

The iterations of semismooth Newton method are showed in Table 1 with $x_0 = -1$. We see that x_k speedily converges to x^* within 6 iterations. Importantly, if x_0 closes 0 or 1 then x_k converges to its, respectively.

Table 1. *The iterations of semismooth Newton method with* $x_0 = -1$

k	*xk*	*G*	*G'*
0	-1	-2	3
1	-0,333333333	-0,592592593	1,3333333
2	0,111111111	0,098765432	0,7777778
3	-0,015873016	-0,031738033	1,9984883
4	8,00455E-06	8,00448E-06	0,999984
5	-6,40738E-11	-1,28148E-10	2
6	0	0	1

Example 5.2. Find the fixed point of function $F(x) = \max\{x^2, x^3, x^4\}$.

Denote $G(x) = x - F(x)$. Noticeably, $x^* = 0$ and $x^* = 1$ are zero-point of $G(x)$. As well as Example 5.1, we apply semismooth Newton method that is given by

$$x_{n+1} = x_n - \frac{1}{G'(x_n)} G(x_n).$$

The iterations of semismooth Newton method are showed in Table 2 with $x_0 = 1,2$. We see that x_k speedily converges to x^* within 5 iterations. Importantly, if x_0 closes 0 or 1 then x_k converges to its, respectively.

Table 2. *The iterations of semismooth Newton method with* $x_0 = 1,2$

k	*xk*	*G*	*G'*
0	1,2	-0,8736	-5,912
1	1,052232747	-0,225585556	-4,6431625
2	1,003648284	-0,014673094	-4,0438593
3	1,000019796	-7,91882E-05	-4,0002376
4	1,000000001	-2,35129E-09	-4
5	1	0	-1
6	1	0	-1

6. Conclusion

In this paper, we have investigated Newton differentiable of function

$$F(x) = \max \; f_1(x), f_2(x), \ldots, f_n(x)$$

in special case $n = 2$ as well as in general case. The sufficient condition of each case is also different. That is, the condition in the case $n = 2$ is weaker than one in general case.

Next, semismooth Newton method is applied to find fixed point for function $F(x)$, the convergence and the quadratic rate of convergence for this problem. All examples have slightly showen that semismooth Newton method reach fast convergence within several iterations to reaches exact solution.

REFERENCE

[1] Ravi P Agarwal, Maria Meehan, and Donal O'Regan. *Fixed point theory and applications*, volume141. Cambridge university press, 2001.

[2] Vasile Berinde. *Iterative approximation of fixed points*, volume 1912. Springer, 2007.

[3] Kim C Border. Fixed point theorems with applications to economics and game theory. Cambridge university press, 1989.

[4] Andrzej Granas and James Dugundji. *Fixed point theory*. Springer Science & Business Media, 2013.

[5] Duong Xuan Hiep, Pham Quy Muoi, Phan Duc Tuan. Some basic properties of Newton derivatives of one variable function. *Journal of Science and Technology, The University of Danang*, Number 9 (118):94-98, 2017.

[6] Mohamed A Khamsi and William A Kirk. *An introduction to metric spaces and fixed point theory (*volume 53). John Wiley & Sons, 2011.

[7] Pham Quy Muoi, Dinh Nho Hào, Peter Maass, and Michael Pidcock. Semismooth newton and quasi-newton methods in weighted l_1 -regularization. *Journal of Inverse and Ill-Posed Problems*, 21(5):665–693, 2013.

Pham Quy Muoi, Dinh Nho Hào, Peter Maass, and Michael Pidcock. Descent gradient methods for non-smooth minimization problems in ill-posed problems. *Journal of Computational and Applied Mathematics*, 298:105–122, 2016.

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