# Computation Initial Degree and Waldschmidt Constant for Sets of Small Number of Multiple Points

Nguyen Chanh Tu*

**Abstract**—Waldschmidt constant is firstly introduced by Waldschmidt in 1975. Since then, many results of this constant was achieved mostly about finding lower bounds. That is recently one of active, fascinating and important topics. However, computation of Waldschmidt constant or the initial degree is very hard in general, even for cases of small numbers of points in the projective plane. Recently, the constants were computed for certain sets of points with one, two and three supporting lines. The paper shows values of the initial degree and Waldschmidt constant for sets with at most 6 points in all configurations in projective plane. These constants represent the complexity of optimal solutions in repeated path problems that have many applications in computer science, informatics theory and telecommunications.

**Index Terms**—Waldschmidt constant, initial degree, zero-dimensional scheme, fat points, path problem

## 1. Introduction

We denote by $\mathbb{P}^n$ the projective space over an algebraically closed field $k$. Let $P \in \mathbb{P}^n$, we say that a form $f$ of the polynomial ring $R := k[x_0, \ldots, x_n]$ has multiplicity at least $m$ at $P$ if all partial derivatives of $f$ of order $< m$ vanishing at $P$.

Let $X := \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$, let $m_1, \ldots, m_s$ be positive integers. Let $\mathcal{P}_i \subset R$ be the defining ideal of $P_i$ consisting of all forms vanishing at $P_i$, for $1 \le i \le s$. We denote by $Z := m_1P_1 + \cdots + m_sP_s$ the zero-dimensional scheme corresponding to the ideal $J = \cap_{i=1}^{s} \mathcal{P}_i^{m_i}$ consisting of all forms of $R$ vanishing at $P_i$ with multiplicity at least $m_i$, for $i = 1, \ldots, s$. This zero-dimensional scheme is called a *fat point scheme*.

Let $A = \oplus_t A_t$, be any homogeneous ideal in $R := k[x_0, \ldots, x_n]$. The value $\alpha(A) = \min\{t | A_t \neq 0\}$ is called the initial degree of $A$. For the ideal $J = \cap_{i=1}^{s} \mathcal{P}^{m_i}$, the initial degree $\alpha(J)$ is the least degree of the hypersurfaces containing $P_i$ with multiplicity at least $m_i$, for $1 \le i \le r$.

***Definition 1.*** Let $X = \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$, and $I = \cap_{i=1}^{s} \mathcal{P}_i \subset R = k[x_0, \ldots, x_n]$. For $m \in \mathbb{N}$, denote $I^{(m)} = \cap_{i=1}^{s} \mathcal{P}_i^m$, the ideal of the fat point scheme $Z = \sum_{i=1}^{s} mP_i$ with equal multiplicity $m$. The value

$$\lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$$

is called the Waldschmidt constant of $I$ or of the set $X$ and denoted by $\gamma(I)$ or $\gamma(X)$.

*Nguyen Chanh Tu is with Falculty of Advanced Science and Technology, The University of Danang - University of Science and Technology, Vietnam.*

**Corresponding author: Nguyen Chanh Tu (e-mail: snctu@dut.udn.vn)*
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It is not hard to see the following basic properties.

***Lemma 1.*** With notations as above, then

1) $\alpha(I^{(m)}) \le m\,\alpha(I)$.
2) $\gamma(I)$ is well defined and $1 \le \gamma(I) \le \dfrac{\alpha(I^{(m)})}{m} \le \alpha(I)$, $\forall m \ge 1$.
3) $\gamma(I) \le \sqrt[n]{s}$.

***Proof 1.*** See [3] and [18].

The constant is firstly introduced by Waldschmidt [22], [23]. Since then, many results of this constant was achieved mostly about finding lower bounds, see [2], [4]–[9], [11], [12], [16], [17]. That is recently one of active, fascinating and important topics as many applications in various areas of mathematics and other sciences, see [18] for more information. These constants represent the complexity of optimal solutions in repeated path problems that have many applications in computer science, informatics theory and telecommunications.

However, computation of $\alpha(I^{(m)})$ and $\gamma(I)$ is very hard in general, even for cases of small numbers of points in the projective plane. For a set of small number of points in general position in $\mathbb{P}^2$, the Waldschmidt constant was known only for cases of $s$ points where $1 \le s \le 9$ or $s$ is a perfect square, see [14], [15]. Recently, the constants were computed for certain sets of $r + s$ points where there are $r$ collinear points and $s$ points in general position, $1 \le s \le 7$, see [21]. Note that, if $s = 1$, it is the case of almost collinear as in [13]. The constants are also computed for certain sets with one, two and three supporting lines as in [18]–[20].

In this paper, we will compute the constants for sets consisting at most 6 points with all possible configuration. We used many tools in computer algebra systems to support of computations.

For proofs in next section, we need to use following results of Bezout.

***Theorem 1 ( [10], I.7.7).*** Let $Y$ be a variety of dimension at least 1 in $\mathbb{P}^n$, and let $H$ be a hypersurface not containing $Y$. Let $Z_1, \ldots, Z_s$ be the irreducible components of $Y \cap H$. Then

$$\sum_{j=1}^{s} i(H, Y; Z_j) \deg Z_j = (\deg Y)(\deg H).$$

Note that $i(H, Y; Z_j)$ is the intersection multiplicity if $Y$ and $H$ along $Z_j$.

***Corollary 1 (Bézout's Theorem, [10], I.7.8).*** Let $Y, Z$ be distinct curves in $\mathbb{P}^2$ having degrees $d, e$. Let $Y \cap Z = \{P_1, ..., P_s\}$. Then

$$\sum_{j=1}^{s} i(H, Z; P_j) = de.$$

Note that $i(H, Z; P) \geq \text{mult}_P(H) \cdot \text{mult}_P(Z)$ and the equality holds if and only if $H$ and $Z$ have no tangent in common at $P$, see [3].

## 2. Main results

Let $X = \{P_1, \ldots, P_s\}$ be a set of points in $\mathbb{P}^2$, let $I = \cap_{i=1}^{s} \mathcal{P}_i \subset R = k[x, y, z]$ be the corresponding ideal of $X$. For $m \in \mathbb{N}$, denote $I^{(m)} = \cap_{i=1}^{s} \mathcal{P}_i^m$.

The paper shows the constants $\alpha(I^{(m)})$ and $\gamma(I) = \gamma(X)$ for all possible configurations of $X$ when $1 \leq s \leq 6$.

First of all, we need a simple lemma for collinear case.

***Lemma 2.*** If all points of $X$ are collinear, then $\alpha(I^{(m)}) = m$ for $m \in \mathbb{N}$ and $\gamma(X) = 1$.

The proof is trivial. From that we state the result for 1 or 2 points as special cases.

***Corollary 2.*** If $X$ consists of 1 or 2 points then $\alpha(I^{(m)}) = m$ for $m \in \mathbb{N}$ and $\gamma(X) = 1$.

We consider now the case of 3 points, which has 2 configurations.

***Theorem 2.*** Let $X$ be the set of three points. If the points are collinear then $\alpha(I^{(m)}) = m$ for $m \in \mathbb{N}$ and $\gamma(X) = 1$. If the points are in general position then $\alpha(I^{(2m)}) = 3m, \alpha(I^{(2m-1)}) = 3m-1$ for $m \in \mathbb{N}$ and $\gamma(X) = 3/2$.

***Proof 2.*** The collinear case follows Lemma 2. For the case of general position, let $f_1, f_2, f_3$ be linear forms defining 3 lines each of them connects 2 points of $X$. It is easy to see that $f = f_1^m f_2^m f_3^m \in I_{3m}^{(2m)}$ and $g = f_1^m f_2^m f_3^{m-1} \in I_{3m-1}^{(2m-1)}$. Then $\alpha(I^{(2m)}) = 3m, \alpha(I^{(2m-1)}) = 3m - 1$ for $m \in \mathbb{N}$ and $\gamma(X) = 3/2$. Note that this result firstly obtained by Nagata in 1959 (see [14]), also mentioned in [18].

The case of 4 points has 3 configurations and the values of the contants are as follows.

***Theorem 3.*** Let $X$ be a set of 4 points in the projective plane.

1) If the points are collinear then $\alpha(I^{(m)}) = m$ for $m \in \mathbb{N}$ and $\gamma(X) = 1$.
2) If there are exactly 3 points of $X$ collinear then $\alpha(I^{(3m)}) = 5m$ for $m \in \mathbb{N}$ and $\gamma(X) = 5/3$.
3) If the points are in general postion then $\alpha(I^{(m)}) = 2m$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.

***Proof 3.***

1) The case of collinear follows Lemma 2.
2) Let $X = \{P_1, P_2, P_3, P_4\}$ where $P_1, P_2, P_3$ are collinear; let $f_0$ be the linear form defining the line containing $P_1, P_2, P_3$; let $f_1, f_2, f_3$ be linear form defining the lines respectively connecting $P_4$ with $P_1, P_2, P_3$. Consider $f = f_0^{2m} f_1^m f_2^m f_3^m$. It is clear that $f \in I_{5m}^{(3m)}$ and $I_{5m-1}^{(3m)} = 0$ for $m \in \mathbb{N}$. Therefore $\alpha(I^{(3m)}) = 5m$ for $m \in \mathbb{N}$ and $\gamma(X) = 5/3$.
3) For the case of general position, let $C$ be the quadratic form defining the conic containing 4 points of $X$. It is clear that $f = C^m \in I_{2m}^{(m)}$ and $I_{2m-1}^{(m)} = 0$. Therefore $\alpha(I^{(m)}) = 2m$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.

The case of 5 points has 4 configurations and the values of the constants are as follows.

***Theorem 4.*** Let $X$ be a set of 5 points in the projective plane.

1) If the points are collinear then $\alpha(I^{(m)}) = m$ for $m \in \mathbb{N}$ and $\gamma(X) = 1$.
2) If there are exactly 4 points collinear then $\alpha(I^{(4m)}) = 7m$ for $m \in \mathbb{N}$ and $\gamma(X) = 7/4$.
3) If there are at most 3 points collinear (including the case of general position), then $\alpha(I^{(m)}) = 2m$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.

***Proof 4.***

1) The case of collinear follows Lemma 2.
2) Let $X = \{P_1, \ldots, P_5\}$ where $P_1, P_2, P_3, P_4$ are collinear; let $f_0$ be the linear form defining the line containing $P_1, P_2, P_3, P_4$; let $f_1, f_2, f_3, f_4$ be linear forms defining the lines respectively connecting $P_5$ with $P_1, P_2, P_3, P_4$. Consider $f = f_0^{3m} f_1^m f_2^m f_3^m f_4^m$. It is clear that $f \in I_{7m}^{(4m)}$ and $I_{7m-1}^{(4m)} = 0$ for $m \in \mathbb{N}$. Therefore $\alpha(I^{(4m)}) = 7m$ and $\gamma(X) = 7/4$.
3) Let $C$ be the quadratic form defining the conic containing 5 points of $X$. It is clear that $f = C^m \in I_{2m}^{(m)}$ and $I_{2m-1}^{(m)} = 0$. Therefore $\alpha(I^{(m)}) = 2m$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.

The case of 6 points has 10 configurations and the values of the constants are as follows.

***Theorem 5.*** Let $X$ be a set of 6 points in the projective plane.

1) If the points are collinear then $\alpha(I^{(m)}) = m$ for $m \in \mathbb{N}$ and $\gamma(X) = 1$.
2) If there are exactly 5 points collinear then $\alpha(I^{(5m)}) = 9m$ for $m \in \mathbb{N}$ and $\gamma(X) = 9/5$.

3) If all points are on an irreducible conic or on 2 lines, each of which contains at most 4 points of $X$, then $\alpha(I^{(m)}) = 2m$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.
4) If all points are on 4 lines, each line contains exactly 3 points of $X$ then $\alpha(I^{(2m)}) = 4m, \alpha(I^{(2m+1)}) = 4m + 3$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.
5) If all points are on 3 lines, each of the lines contains exactly 3 points of $X$, then $\alpha(I^{(4m)}) = 9m$ for $m \in \mathbb{N}$ and $\gamma(X) = 9/4$.
6) If there are 3 points collinear and $X$ can not contained in 2 lines then $\alpha(I^{(3m)}) = 7m$ for $m \in \mathbb{N}$ and $\gamma(X) = 7/3$.
7) If the points are in general position then $\alpha(I^{(5m)}) = 12m$ for $m \in \mathbb{N}$ and $\gamma(X) = 12/5$.

***Proof 5.***

1) Follows Lemma 2.
2) Let $X = \{P_1, \ldots, P_5, Q\}$ such that $P_1, \ldots, P_5$ lies on a line $l = V(f_0)$, point $Q \notin l$. Let $f_1, \ldots, f_5$ be linear forms defining the lines respectively connecting $Q$ with $P_1, \ldots, P_5$. Consider $f = f_0^{4m} f_1^m f_2^m f_3^m f_4^m f_5^m$. It is clear that $f \in I_{9m}^{(5m)}$ and $I_{9m-1}^{(5m)} = 0$ for $m \in \mathbb{N}$. This follows that $\alpha(I^{(5m)}) = 9m$ for $m \in \mathbb{N}$ and $\gamma(X) = 9/5$.
3) If $X$ lies on a conic defined by a quadratic form $C$ or if $X$ are on 2 lines defined by linear forms $f_1, f_2$, then it is clear that $C^m$ or $f_1^m f_2^m$ are in $I_{2m}^{(m)}$, and $I_{2m-1}^{(m)} = 0$. That implies that $\alpha(I^{(m)}) = 2m$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.
4) Let $X = \{P_1, \ldots, P_6\}$ such that there are 4 lines defined by linear forms $l_1, l_2, l_3, l_4$ containing all points of $X$ as in Fig. 1. It is clear that $l_1^m l_2^m l_3^m l_4^m \in I_{4m}^{(2m)}$ and $l_1^{m+1} l_2^{m+1} l_3^{m+1} l_4^m \in I_{4m+3}^{(2m+1)}$. Then $\alpha(I^{(2m)}) = 4m, \alpha(I^{(2m+1)}) = 4m + 3$ for $m \in \mathbb{N}$ and $\gamma(X) = 2$.

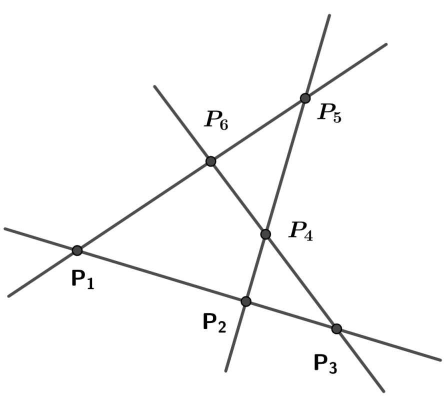


Fig. 1: 6 points on 4 lines

5) Let $X = \{P_1, \ldots, P_6\}$ such that there are 3 lines defined by linear forms $l_1, l_2, l_3$ containing all points of $X$ as in Fig. 2. Note that, three points $P_2, P_4, P_6$ are not collinear. Let $\overline{P_2P_4} = V(d_1), \overline{P_2P_6} = V(d_2), \overline{P_4P_6} = V(d_3)$. We see that $l_1^{2m} l_2^{2m} l_3^{2m} d_1^m d_2^m d_3^m \in I_{9m}^{(4m)}$, thus

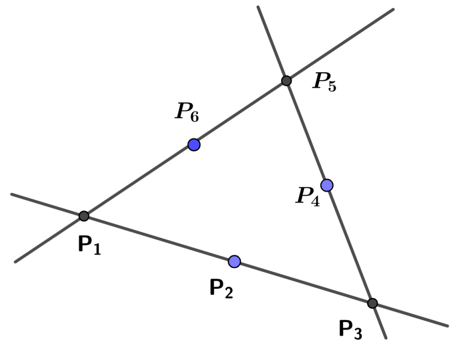


Fig. 2: 6 points on 3 lines

$\alpha(I^{(4m)}) \leq 9m$ for all $m \geq 1$ and $\gamma(I) \leq 9/4$.
We will show that $I_{9m-1}^{(4m)} = 0$ for all $m \geq 1$. Suppose that there exists $0 \neq f \in I_{9m-1}^{(4m)}$. By Bézout's Theorem, we see that $l_1 \mid f$ and similarly $l_2 \mid f, l_3 \mid f$. We can write $f = l_1^a l_2^b l_3^c g$, where $\deg(g) = d$ and $a+b+c+d = 9m-1$. If $d_1$ is not a factor of $f$, then by Bézout's Theorem, we have $a + b + d \geq 8m$, implying $c < m$. Similarly we have $a < m, b < m$. Note that $a + b \geq 4m, b + c \geq 4m, a + c \geq 4m$, then $a + b + c \geq 6m$. This contradicts to $a < m, b < m, c < m$. Therefore $d_1, d_2, d_3$ are factors of $f$. Moreover, we see that $a \geq 2, b \geq 2, c \geq 2$. It means that $f$ has a factor $f_0 = l_1^2 l_2^2 l_3^2 d_1 d_2 d_3$ of degree 9 Let $f_1 = f/f_0$, then $f_1 \in I_{9(m-1)-1}^{(4(m-1))}$. By induction, it is impossible since when $m = 1$ any member in $I_8^{(4)}$ can not have a factor of degree 9.
Thus $I_{9m-1}^{(4m)} = 0$ for all $m \geq 1$. It means that $\alpha(I^{(4m)}) = 9m$ for all $m \geq 1$ and $\gamma(I) = 9/4$.
6) There are 2 configurations for this case as in Fig. 3. For configuration (b), where three points

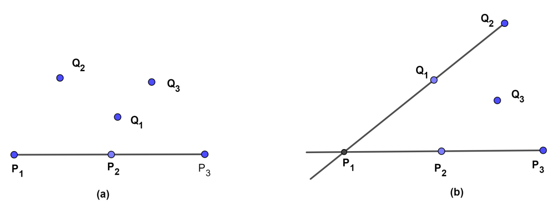


Fig. 3: 6 points with at least one supporting line

$P_1, P_2, P_3$ lie on a line $V(l_1)$, three points $P_1, Q_1, Q_2$ lie on a line $V(l_2)$. Let $C = V(g)$ be the conic containing $P_2, P_3, Q_1, Q_2, Q_3$ and $d$ be the linear form defining the line $\overline{P_1Q_3}$. Then $l_1^m l_2^m d^m g^{2m} \in I_{7m}^{(3m)}$ and $I_{7m-1}^{(3m)} = 0$. It means that $\alpha(I^{(3m)}) = 7m$ for $m \in \mathbb{N}$ and $\gamma(X) = 7/3$.

For configuration (a), where there is a line $V(l)$ containing $P_1, P_2, P_3$. Let $C_1, C_2, C_3$ be quadratic forms defining 3 conics, each of which contains $Q_1, Q_2, Q_3$ and 2 of $P_1, P_2, P_3$. Then $C_1C_2C_3l \in I_7^{(3)}$. It means that $\alpha(I^{(3)}) \leq 7$ and $\gamma(X) \leq 7/3$.

Suppose that there exists $f \in I^{(3m)}_{7m-1}$. On the line $l$, the polynomial $f$ vanishes at each point $P_1, P_2, P_3$ with multiplicities at least $3m$, there for $f = l^a g$ where $l \nmid g, \deg(g) = b$ and $a + b = 7m - 1$. We have $b \geq 3m$.

If any of $\overline{Q_1Q_2} = V(l_1), \overline{Q_2Q_3} = V(l_2), \overline{Q_1Q_3} = V(l_3)$ is not a divisor of $V(g)$, then $b \geq 6m$. On the other hands, we have $3a+b \geq 3 \cdot 3m$. Then $3a+3b \geq 9m+12m = 21m$. Therefore $a + b \geq 7m$, this contradicts to $a + b = 7m - 1$. If $g = l_1^{b_1} l_2^{b_2} l_3^{b_3} h$, where $\deg(h) = c$ and $l_i \nmid h$ for any $1 \leq i \leq 3$. Then $b_1+b_2+b_3+c = b$ and $2b_1+2b_2+2b_3+c \geq 3 \cdot 3m$. On the other hands $3a + c \geq 9m$. This implies that $3(b_1 + b_2 + b_3) + 2c \geq 9m + b \geq 12m$. Then $3a + 3(b_1 + b_2 + b_3 + c) = 3(a + b) \geq 21m$ and $a + b \geq 7m$, this contradicts to $a + b = 7m - 1$. It means that $\alpha(I^{(3m)}) = 7m$ and $\gamma(I) = 7/3$.

7) Let $X$ consist of 6 points in general position, let $C_1, C_2, C_3, C_4, C_5, C_6$ be conics containing 5 of 6 points in $X$. Denote $C_i = V(f_i)$ for $1 \leq i \leq 6$. Then the curve $f = f_1^m \cdots f_6^m \in I^{(5m)}_{12m}$ and $I^{(5m)}_{12m-1}$. This implies that $\alpha(I^{(5m)}) = 12m$ for all $m \geq 1$ and $\gamma(I) = 12/5$.

## 3. Conclusion

The paper shows values and detail computations of the initial degree and Waldschmidt constant for sets with at most 6 points in all configurations in projective plane. The methods of computation in the paper can be extended for more complicated configurations in projective plane. These constants represent the complexity of optimal solutions in repeated path problems that have many applications in computer science, informatics theory and telecommunications.

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**Nguyen Chanh Tu** received his B.S (1990) at Hue University, M.S. (1995) at Mathematical Research Institute of The Netherlands and Ph.D. (2000) degree at Utrecht University, The Netherlands. In 2007, he got the title of Associate Professor. Now, he is a lecturer of The University of Danang - University of Science and Technology, Vietnam. His current research interest lies in Computer Algebra, Algebraic Geometry, Singularity Theory.