METHOD ORDERABLE CASCADE DECOMPOSITION FOR CONTROL SYSTEM

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Abstract - In this paper, we propose a method to determine the state function $y(x,t)$ and control function $u(t,x)$ of the dynamical system represented by a system of equations with first-order partial derivatives $\frac{\partial y}{\partial t} = B\frac{\partial y}{\partial x} + Du(t,x)$ under boundary and the first-order partial derivatives conditions of state function where B, D are real matrices with corresponding sizes. The basis of the theory is a method to prove the orderable cascade decomposition to transform the original system into the equivalent system in the type $\frac{\partial y_p}{\partial t} = B\frac{\partial y_p}{\partial x} + Du_p(t,x)$. In the final step, we obtain the state function $y(t,x)$ satisfying the conditions and substituting this in the previous step. Continuing this process, we can find out the state function $y(t,x)$ and controllable function $u(t,x)$ of the original dynamical system.

Key words - Control function; state function; descriptor system; method cascade; differential equation

1. Introduction

Consider the system:

$$\frac{\partial y}{\partial t} = B\frac{\partial y}{\partial x} + Du(t,x), \quad t \in [0,T], x \in [0,x_k], \tag{1}$$

Where, $y = y(t,x) \in \mathbb{R}^n$, and B, D are matrices of corresponding sizes.

Definition 1.1. *System* (1) *is called completely controllable if there exists a control function* $u(t,x)$ *under its influence the system* $u(t,x)$ *can transform from an arbitrary initial state:*

$$y(0,x) = \alpha(x) \in \mathbb{R}^n, \tag{2}$$

and

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \alpha^*(x) \in \mathbb{R}^n, \tag{3}$$

to final state

$$y(T,x) = \beta(x) \in \mathbb{R}^n, \tag{4}$$

and

$$\left.\frac{\partial y}{\partial t}\right|_{t=T} = \beta^*(x) \in \mathbb{R}^n, \tag{5}$$

in the period $[0,T]$, *for all* $T > 0$.

The requirement for smoothies of the functions $\alpha(x), \beta(x), \alpha^*(x), \beta^*(x)$ can be a necessary condition for the controllability of the system. For example, if:

$$\frac{\partial y_1(t,x)}{\partial t} = u(t,x), \qquad \frac{\partial y_2(t,x)}{\partial t} = \frac{\partial y_1(t,x)}{\partial t},$$

$$\alpha(x) = (\alpha_1(x), \alpha_2(x)), \beta(x) = (\beta_1(x), \beta_2(x)),$$

and $\alpha^*(x) = (\alpha_1^*(x), \alpha_2^*(x)), \beta^*(x) = (\beta_1^*(x), \beta_2^*(x))$, then the second equation has the form:

$$\left.\frac{\partial^2 y_2(t,x)}{\partial t^2}\right|_{t=0} = \left.\frac{\partial \alpha_1^*(x)}{\partial t}\right|_{t=0};$$

$$\left.\frac{\partial^2 y_2(t,x)}{\partial t^2}\right|_{t=T} = \left.\frac{\partial \beta_1^*(x)}{\partial t}\right|_{t=0},$$

therefore, the differentability of $\alpha^*(x)$ and $\beta^*(x)$ are truly necessary.

We pose the issue of needing to clarify the requirements for matrices B and D so that if those factors are satisfied, system (1) is completely controllable, the problem of building systems $y(t,x)$ and $u(t,x)$ in the form of analytic functions and determining the smoothness of functions $\alpha(x), \beta(x), \alpha^*(x), \beta^*(x)$ is a sufficient condition for the control problem. To solve the problems posed by control systems with partial derivatives, we introduce the orderable cascade decomposition method. Solving the system (1) with the conditions (2), (3), (4), (5), we will consider the case where B and D are real matrices.

The fact that the function $y(t,x)$ will be constructed first and then the function $u(t,x)$ will be calculated according to the orderable decomposition method, i.e the real dynamical system is described by the system (1) with a predict control function $u(t,x)$ and initial state functions (2) and (3), it is necessary to obtain the state $\beta(x)$ and $\beta^*(x)$ respectively at time $t = T$.

The orderable decomposition method is based on representing space into subspaces. In practice, we need to include symbols in each change process. In this paper, we will describe the obtained results through an example.

2. Initial steps

We use the following property of the operator $G: \mathbb{R}^m \to \mathbb{R}^n$ as follows:

$$\mathbb{R}^m = CoimG + KerG$$
$$\mathbb{R}^n = ImG + CokerG \tag{6}$$

Where, $KerG$ is called kernel of G, $Im\,G$ is called image of G, $CoimG$ and $CokerG$ are complement of $KerG$ và ImG respectively in $\mathbb{R}^m$ and $\mathbb{R}^n$. The contraction matrix $\tilde{G}$ of G on $CoimG$ is a inverse matrix $\tilde{G}^{-1}$. The projection onto subspaces $Ker\,G$ and $Coker\,G$ are denoted by $P(G)$ as $Q(G)$ respectively. The operator $\tilde{G}^{-1}(I - Q(G))$ is called inverse matrix of G and is denoted by G^- (Later, we will denote the identity operators in the respective spaces). In the particular case, $G^-G = I - P(G)$ and $GG^- = I - Q(G)$.

Note that ai at the decomposition step (6), projections $P(G)$, $Q(G)$ and a operater G^- can be written in different forms depending on the dependency of the choice of basis in $KerG$ and $CokerG$ or in dependence on the numbering of base elements. Next we use the following the lemma.

Lemma 2.1: *(see [3]) The equation:*

$$Gv = w, v \in \mathbb{R}^m, w \in \mathbb{R}^n \tag{7}$$

is equivalent to the system:

$$\begin{cases} Q(G)w = 0 \\ v = G^- w + P(G)z \end{cases} \tag{8}$$

for all $P(G)z \in KerG, P(G)z = P(G)v$.

Example 2.2. Consider the system of equations:

$$\begin{cases} v_1 - 2v_2 = w_1 \\ v_1 = w_2 \\ 2v_2 = w_3, \end{cases} \tag{9}$$

is equivalent to the system:

$$\begin{cases} w_2 - w_3 = w_1 \\ v_1 = w_2 \\ 2v_2 = w_3, \end{cases}$$

or
$$\begin{cases} w_1 - w_2 + w_3 = 0 \\ v_1 = w_2 \\ 2v_2 = w_2 - w_1. \end{cases}$$

In system (9) matrix G is defined by:

$$G = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We have: $KerG = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}; ImG = \left\{\begin{bmatrix} b-c \\ b \\ c \end{bmatrix} : b, c \in \mathbb{R}\right\}$.

All elements $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ can be represented as form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in KerG, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in CoimG$.

All elements $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$ can be represented as form:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} w_2 - w_3 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} w_1 - w_2 + w_3 \\ 0 \\ 0 \end{bmatrix},$$

with $\begin{bmatrix} w_2 - w_3 \\ w_2 \\ w_3 \end{bmatrix} \in ImG, \begin{bmatrix} w_1 - w_2 + w_3 \\ 0 \\ 0 \end{bmatrix} \in CokerG$.

We proceed to find projections $P(G)$ and $Q(G)$ onto $Ker\ G$ and $Coker\ G$ rexpectively. We have:

$$P(G)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow P(G) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$Q(G)\begin{bmatrix} w_2 - w_3 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} w_1 - w_2 + w_3 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow Q(G) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G^- \begin{bmatrix} u_2 - u_3 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow G^- = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Decomposition steps for control systems

By Lemma (6), system (1) is equivalent to:

$$\begin{aligned} &Q(D)\frac{\partial y}{\partial t} = Q(D)B\frac{\partial y}{\partial x}, \\ &u(t,x) = D^-\left(\frac{\partial y}{\partial t} - B\frac{\partial y}{\partial x}\right) + P(D)z(t,x), \end{aligned} \tag{10}$$

for all $P(D)z(t,x) \in KerD$.

The second equation of system (10) allows us to determine the control function $u(t,x)$, then from the first equation of the above system, we can determine the function $y(t,x)$ that satisfies the conditions from (2) to (5).

We denote:

$$D = D_0, B = B_0, Q(D) = Q_0, P(D) = P_0,$$
$$D^- = D_0^-, y(t,x) = y_0, u(t,x) = u_0.$$

Using the property $(I - Q_0) = (I - Q_0)^2$ of the projection $Q_0 = Q_0^2$, the first equation of the system (10) is rewritten as follows:

$$Q_0\frac{\partial y_0}{\partial t} = Q_0 B_0 \frac{\partial y_0}{\partial x}, \tag{11}$$

and converted as:

$$\begin{aligned} \frac{\partial Q_0 y_0}{\partial t} &= Q_0 B_0 Q_0 \frac{\partial Q_0 y_0}{\partial x} \\ &+ Q_0 B_0 (I - Q_0)\frac{\partial (I-Q_0) y_0}{\partial x}. \end{aligned} \tag{12}$$

We put in the following notations:

$$\begin{aligned} &Q_0 y_0 = y_1, (I - Q_0) y_0 = u_1, \\ &Q_0 B_0 Q_0 = B_1, Q_0 B_0 (I - Q_0) = D_1 \end{aligned} \tag{13}$$

then the equation (3.4) has form:

$$\frac{\partial y_1}{\partial t} = B_1\frac{\partial y_1}{\partial x} + D_1\frac{\partial u_1}{\partial x} \tag{14}$$

The final equation is distinct from the initial equation. First, at the derivative of the quasi-control function $u_1 = u_1(t,x)$, The second is that the resulting equation is contained in $CokerD$ space with a reduced number of equations and is less than the original system. And third, here $y_1(t,x)$ and $u_1(t,x)$ must satisfy the conditions from (2) to (5) and (14):

$$y_1(0,x) = Q_0\alpha(x),$$
$$y_1(T,x) = Q_0\beta(x),$$
$$u_1(0,x) = (I - Q_0)\alpha(x),$$
$$u_1(T,x) = (I - Q_0)\beta(x),$$
$$\left.\frac{\partial y_1}{\partial t}\right|_{t=0} = Q_0\alpha^*(x),$$
$$\left.\frac{\partial y_1}{\partial t}\right|_{t=T} = Q_0\beta^*(x).$$

Using (13), we substitute the above conditions as follows:

$$\begin{aligned} &y_1(0,x) = Q_0\alpha(x) \coloneqq \alpha_1^0(x), \\ &y_{1(T,x)} = Q_0\beta(x) : = \beta_1^0(x) \end{aligned} \tag{15}$$

$$\left.\frac{\partial y_1}{\partial t}\right|_{t=0} = Q_0\alpha^*(x) := \alpha_1^{*1}(x),$$
$$\left.\frac{\partial y_1}{\partial t}\right|_{t=T} = Q_0\beta^*(x) = \beta_1^{*1}(x) \quad (16)$$

$$\left.\frac{\partial^2 y_1}{\partial t^2}\right|_{t=0} = Q_0B_0\frac{\partial}{\partial t}\left(\frac{\partial y_1}{\partial x}\right)\Big|_{t=0} = Q_0B_0\frac{\partial}{\partial x}\left(\frac{\partial y_1}{\partial t}\right)\Big|_{t=0} = Q_0B_0\alpha_1^{*1}(x) := \alpha_1^{*2}(x). \quad (17)$$

$$\left.\frac{\partial^2 y_1}{\partial t^2}\right|_{t=T} = Q_0B_0\frac{\partial}{\partial t}\left(\frac{\partial y_1}{\partial x}\right)\Big|_{t=T} = Q_0B_0\frac{\partial}{\partial x}\left(\frac{\partial y_1}{\partial t}\right)\Big|_{t=T} = Q_0B_0\beta_1^{*1}(x) := \beta_1^{*2}(x). \quad (18)$$

here, the above indices correspond to the order of the derivative. Thus, we receive:

Lemma 3.1. *The system from* (1) *to* (5) *are equivalent to the system* (10) *and from* (15) *to* (18).

4. Iterative progress of decomposition steps

In the second step of the decomposition process, we construct the equation (14) with the conditions from (15) to (18). Here, $D_1: ImD_0 \to CokerD_0$ is a finite dimensional operator, we have the decomposition as follows:

$$ImD_0 = CoimD_1 + CokerD_1; \quad (19)$$
$$CokerD_0 = ImD_1 + CokerD_1,$$

with $\dim CokerD_0 = n_1, \dim CokerD_1 = n_2$, $n \geq n_1 \geq n_2$.

In the case $n_1 = n_2$, we have $ImD_1 = \{0\}$ and the system:

$$\frac{\partial y_1}{\partial t} = B_1\frac{\partial y_1}{\partial x},$$

with the conditions from (15) to (18) is impossible to solve. Therefore, the system (1.1) is not controllable. Then, we assume that $n > n_1 > n_2 > \cdots$ Using (19), we have the transformation step from the equation (14) to the new systems in subspace $CoimD_1$:

$$\frac{\partial u_1}{\partial x} = D_1^-\left(\frac{\partial y_1}{\partial t} + B_1\frac{\partial y_1}{\partial x}\right) + P_1z_1(t,x),$$

where, $P_1z_1(t,x) \in KerD_1$ is an arbitrarily vector function, then the equation:

$$Q_1\frac{\partial y_1}{\partial t} = Q_1B_1\frac{\partial y_1}{\partial x}, \quad (20)$$

can be converted to the following equation

$$\frac{\partial y_2}{\partial t} = B_2\frac{\partial y_2}{\partial x} + D_2\frac{\partial u_2}{\partial x}$$

with $u_2 \in ImD_1, y_2 \in CokerD_1$.

We put in the following notations:

$$B_j = Q_{j-1}B_{j-1}Q_{j-1}$$
$$D_j = Q_{j-1}B_{j-1}(I - Q_{j-1}) \quad (21)$$
$$y_j = y_j(t,x) = Q_jy_{j-1};$$
$$u_j = u_j(t,x) = (I - Q_{j-1})y_{j-1}$$

with $n_j = dim\ CokerD_{j-1}, j = 1,2,3,\ldots$ At the j^{th} step, we use 21) and transfer from the equation:

$$\frac{\partial y_{j-1}}{\partial t} = B_{j-1}\frac{\partial y_{j-1}}{\partial x} + D_{j-1}\frac{\partial u_{j-1}}{\partial x}$$

to the system:

$$\frac{\partial u_{j-1}}{\partial x} = D_{j-1}^-\left(\frac{\partial y_{j-1}}{\partial t} + B_{j-1}\frac{\partial y_{j-1}}{\partial x}\right) + P_{j-1}z_{j-1}(t,x), \quad (22)$$

$$Q_{j-1}\frac{\partial y_{j-1}}{\partial t} = Q_{j-1}B_{j-1}\frac{\partial y_{j-1}}{\partial x} \quad (23)$$

for all $P_{j-1}z_{j-1}(t,x) \in KerD_{j-1}$ and from the equation (4.5) we get the following equation:

$$\frac{\partial y_j}{\partial t} = B_j\frac{\partial y_j}{\partial x} + D_j\frac{\partial u_j}{\partial x} \quad (24)$$

Then, $n > n_1 > n_2 > \ldots > n_j > \cdots$ there are only two possibilities $n > n_1 > n_2 > \ldots > n_p = n_{p+1} \neq 0$ or $n > n_1 > n_2 > \ldots > n_p = n_{p+1} = 0$. We have $D\_p = 0$ in the first case and $Q_p = 0$ in the second case. Thus, we have the following statement.

Lemma 4.1. *There exists a natural number* p *such that system* (1) *is equivalent to the system formed by the second equation of system* (10) *and the following relations:*

$$y(t,x) = y_1(t,x) + u_1(t,x), \quad (25)$$

for all $P_jz_j(t,x) \in KerD_j$*, here* D_p *is a surjection* ($Q_p = 0$).

If $D_p = (0)$ then $CokerD_{p-1} = CokerD_p$ and

$$\mathbb{R}^n = \mathrm{ImD} + \mathrm{CokerD} = \mathrm{ImD} + \mathrm{ImD}_1 + \mathrm{CokerD}_1 = \ldots = \mathrm{ImD} + \mathrm{ImD}_1 + \ldots + \mathrm{ImD}_{p-1} + \mathrm{CokerD}_p.$$

If D_p is a surjection, $CokerD_{p-1} = ImD_p$ and $\mathbb{R}^n = ImD + ImD_1 + \ldots + ImD_{p-1} + ImD_p$. Moreover:

$$\begin{aligned} y(t,x) &= y_1(t,x) + u_1(t,x) \\ &= y_2(t,x) + u_2(t,x) + u_1(t,x) \\ &= y_p(t,x) + \sum_{j=1}^{p} u_j(t,x) \\ &= y_p(t,x) + \sum_{j=1}^{p}(I - Q_{j-1})y_{j-1} + P_jz_j(t,x) \end{aligned}$$

for all $P_jz_j(t,x) \in KerD_j$.

5. Boundary condition at each decomposition step

At the second step, we construct the condition of the function $y_2(t,x)$ as follows. From (15):

$$y_2(0,x) = Q_1y_1(0,x) = Q_1\alpha_1^0(x) := \alpha_2^0(x),$$
$$y_2(T,x) = Q_1y_1(T,x) = Q_1\beta_1^0(x) := \beta_2^0(x), \quad (26)$$

$$\left.\frac{\partial y_2}{\partial t}\right|_{t=0} = Q_1\alpha_1^{*1}(x) := \alpha_2^{*1}(x), \quad (27)$$

$$\left.\frac{\partial y_2}{\partial t}\right|_{t=T} = Q_1\beta_1^{*1}(x) \coloneqq \beta_2^{*1}(x),$$
$$\left.\frac{\partial^2 y_2}{\partial t^2}\right|_{t=0} = Q_1B_1\frac{\partial}{\partial t}\left(\frac{\partial y_2}{\partial x}\right)\Big|_{t=0} = Q_1B_1\frac{\partial}{\partial x}\left(\frac{\partial y_2}{\partial t}\right)\Big|_{t=0} = Q_1B_1\alpha_2^{*1}(x) \coloneqq \alpha_2^{*2}(x)$$
$$\left.\frac{\partial^2 y_2}{\partial t^2}\right|_{t=T} = Q_1B_1\frac{\partial}{\partial t}\left(\frac{\partial y_2}{\partial x}\right)\Big|_{t=T} = Q_1B_1\frac{\partial}{\partial x}\left(\frac{\partial y_2}{\partial t}\right)\Big|_{t=0} = Q_1B_1\beta_2^{*1}(x) \coloneqq \beta_2^{*2}(x) \quad (28)$$

From (20) we have:

$$\frac{\partial^3 y_2}{\partial t^3} = Q_1B_1\frac{\partial}{\partial t}\left(\frac{\partial^2 y_2}{\partial x^2}\right) = Q_1B_1\frac{\partial}{\partial x}\left(\frac{\partial^2 y_2}{\partial t^2}\right). \quad (29)$$

From (16) and(29) yielding:

$$\left.\frac{\partial^3 y_2}{\partial t^3}\right|_{t=0} = Q_1B_1\frac{\partial\alpha_2^{*2}(x)}{\partial x} = \alpha_2^{*3}(x),$$
$$\left.\frac{\partial^3 y_2}{\partial t^3}\right|_{t=T} = Q_1B_1\frac{\partial\beta_2^{*2}(x)}{\partial x} = \beta_2^{*3}(x). \quad (30)$$

At the j^{th} composition step we have:

$$y_j(0,x) = Q_{j-1}y_{j-1}(0,x) = Q_{j-1}\alpha_{j-1}^0(x) := \alpha_j^0(x);$$
$$y_j(T,x) = Q_{j-1}y_{j-1}(T,x) = Q_{j-1}\beta_{j-1}^0(x) := \beta_j^0(x);$$
$$\left.\frac{\partial y_j}{\partial t}\right|_{t=0} = Q_{j-1}\alpha_{j-1}^{*}{}^1(x) \coloneqq \alpha_j^{*1}(x),$$
$$\left.\frac{\partial y_j}{\partial t}\right|_{t=T} = Q_{j-1}\beta_{j-1}^{*}{}^1(x) \coloneqq \beta_j^{*1}(x),$$
$$\left.\frac{\partial^2 y_j}{\partial t^2}\right|_{t=0} = Q_{j-1}B_{j-1}\frac{\partial}{\partial t}\left(\frac{\partial y_j}{\partial x}\right)\Big|_{t=0} = Q_{j-1}B_{j-1}\frac{\partial}{\partial x}\left(\frac{\partial y_j}{\partial t}\right)\Big|_{t=0} = Q_{j-1}B_{j-1}\alpha_j^{*1}(x) \coloneqq \alpha_2^{*2}(x);$$
$$\left.\frac{\partial^2 y_j}{\partial t^2}\right|_{t=T} = Q_{j-1}B_{j-1}\frac{\partial}{\partial t}\left(\frac{\partial y_j}{\partial x}\right)\Big|_{t=T} = Q_{j-1}B_{j-1}\frac{\partial}{\partial x}\left(\frac{\partial y_j}{\partial t}\right)\Big|_{t=T} = Q_{j-1}B_{j-1}\beta\,{}_j^{*1}(x) \coloneqq \beta_2^{*2}(x);$$
$$\left.\frac{\partial^3 y_j}{\partial t^3}\right|_{t=0} = Q_{j-1}B_{j-1}\frac{\partial\alpha_j^{*2}(x)}{\partial x} = \alpha_j^{*3}(x),$$
$$\left.\frac{\partial^3 y_j}{\partial t^3}\right|_{t=T} = Q_{j-1}B_{j-1}\frac{\partial\beta_j^{*2}(x)}{\partial x} = \beta_j^{*3}(x).$$

Continuing the above process, we get:

$$\left.\frac{\partial^{j+1} y_j}{\partial t^{j+1}}\right|_{t=0} = Q_{j-1}B_{j-1}\frac{\partial\alpha_j^{*j}(x)}{\partial x} = \alpha_j^{*j+1}(x),$$
$$\left.\frac{\partial^{j+1} y_j}{\partial t^{j+1}}\right|_{t=T} = Q_{j-1}B_{j-1}\frac{\partial\beta_j^{*j}(x)}{\partial x} = \beta_j^{*j+1}(x).$$

Similarly, at the p^{th} decomposition step we have the following conditions:

$$y_p(0,x) \coloneqq \alpha_p^0(x),$$
$$y_p(T,x) \coloneqq \beta_p^0(x), \quad (31)$$
$$\left.\frac{\partial^j y_p}{\partial t^j}\right|_{t=0} = \alpha_p^{*j}(x); \left.\frac{\partial^j y_p}{\partial t^j}\right|_{t=T} = \beta_p^{*j}(x),$$

with $j = 0,\ldots,p$ where:

$$\alpha_p^0(x) = Q_{p-1}\alpha_{p-1}^0(x);$$
$$\beta_p^0(x) = Q_{p-1}\beta_{p-1}^0(x), \quad (32)$$
$$\alpha_p^{*j}(x) = Q_{p-1}B_{p-1}\alpha_p^{*j-1}(x);$$
$$\beta_p^{*j}(x) = Q_{p-1}B_{p-1}\beta_p^{*j-1}(x),$$

where $j = 1,2,\ldots,p$.

Lemma 5.1. *If $\alpha^*(x)$ and $\beta^*(x)$ are differentiable p times, the system $(1)-(5)$ is equivalent to (14), (24), (30).*

Note 5.2. *In the equation (25), D_p is a zero matrix or a surjection.*

If D_p is a zero matrix, the solution of the equation (24) founds as form:

$$\frac{\partial y_p}{\partial t} = B_p\frac{\partial y_p}{\partial x},$$

is not satisfied the condition (30). Therefore, $D_p = (0)$ then system (1) is not controllable. So, the problem $(24)-(30)$ is resolved if D_p is a surjection.

6. Solution of problem at the last step

At the final step of the decomposition process, we receive the equation (24) and the condition (30) for y_p. With $y_p = y_p(t,x)$ we need to find a non-differentiable vector function with respect to t and x satisfying (30). Such as:

$$y_p(t,x) = \sum_{i=1}^{2p+4} c_i(x)\varphi_i(t), \quad (33)$$

Where, $\varphi_i(t)$ are scalar and linearly independent functions, $c_i(x)$ is the vector coefficient found by placing (33) in (30). Furthermore, for each component of the vector function $c_i(x)$ we obtain a linear system with the matrix determinant of the following form: the first $p+1$ rows will be a function of the Wronxki determinant of the functions $\varphi_i(t)$ at time $t=0$, at the next $p+1$ rows will be a function of the Wronxki determinant of the functions $\varphi_i(t)$ at time $t=T$. In the particular case if $\varphi_i(t) = t^{i-1}, i = 1,\ldots,2p+4$ the determinant of the coefficient matrix is non-zero and the vector function $y_p(t,x)$ constructed according to (33) with above function $\varphi_i(t)$ are unique. Obviously, if we choose $\varphi_i(t)$ differently, as long as the term of the coefficient matrix in the received system is non-zero, we will get a different solution $y_p(t,x)$.

Now, we will solve the problem:

$$D_p \frac{\partial u_p}{\partial x} = \frac{\partial y_p}{\partial t} - B_p \frac{\partial y_p}{\partial x} \quad (34)$$

with notice that, $u_p(t,x) = (I - Q_{j-1})y_p(t,x)$ và $y_{p-1}(t,x)$ satisfies the following conditions:

$$\left.\frac{\partial^j y_{p-1}}{\partial t^j}\right|_{t=0} = \alpha^{*j}_{p-1}(x)$$
$$\left.\frac{\partial^j y_{p-1}}{\partial t^j}\right|_{t=T} = \beta^{*j}_{p-1}(x) \quad (35)$$

Integrating both sides of equation (34) with respect to x we get:

$$D_p u_p(t,x) = \int_0^x \left(\frac{\partial y_p}{\partial t} - B_p \frac{\partial y_p}{\partial x}\right) ds + D_p h_p(t), \quad (36)$$

where $h_p(t)$ is some vector function, from there we have:

$$u_p(t,x) = D_p^- \int_0^x \left(\frac{\partial y_p}{\partial t} - B_p \frac{\partial y_p}{\partial x}\right) ds + h_p(t) + P_p z(t,x), \quad (37)$$

where:

$P_p z(t,x) = P_p u_p(t,x) = P_p(I - Q_{j-1})y_p(t,x)$. So:

$$\left.\frac{\partial^j P_p z(t,x)}{\partial t^j}\right|_{t=0} = P_p(I - Q_{j-1})\alpha^{*j}_{p-1}(x), \quad j = 1,2,\dots,p-1. \quad (38)$$

Since $h_p(t) = u_p(t,0) = (I - Q_{j-1})y_p(t,0)$ in (37), we get:

$$\left.\frac{\partial^j z(t,x)}{\partial t^j}\right|_{t=0} = (I - Q_{j-1})\alpha^{*j}_{p-1}(x), \quad j = 1,2,\dots,p-1. \quad (39)$$

Likewise:

$$\left.\frac{\partial^j P_p z(t,x)}{\partial t^j}\right|_{t=T} = P_p(I - Q_{j-1})\beta^{*j}_{p-1}(x), \quad j = 1,2,\dots,p-1,$$
$$\left.\frac{\partial^j z(t,x)}{\partial t^j}\right|_{t=T} = (I - Q_{j-1})\beta^{*j}_{p-1}(x), \quad j = 1,2,\dots,p-1. \quad (40)$$

Finally, $u_p(t,x)$ is defined by formula (37) with $y_p(t,x)$ being a previous build function and $h(t), P_p z(t,x)$ are arbitrary vector functions satisfy the conditions from (38) to (40). Then, at the last step of the decomposition process, under the condition that D_p is a surjection, we get two vector functions $y_p(t,x)$ and $u_p(t,x)$ satisfying (16) and (30).

7. Applicable example

Consider $y(t,x) = \{y_1(t,x), y_2(t,x)\}$, $0 \le t \le 1, 0 \le x \le 1$, where:

$$\begin{cases} \dfrac{\partial y_1(t,x)}{\partial t} = u(t,x) \\ \dfrac{\partial y_2(t,x)}{\partial t} = \dfrac{\partial y_1(t,x)}{\partial t} + \dfrac{\partial y_2(t,x)}{\partial x}. \end{cases} \quad (41)$$

We set the requirement to determine $u(t,x)$ such that at $t = 0$ they satisfy the following conditions:

$$y_1(0,x) = 2x^2, \quad y_2(0,x) = e^x,$$
$$\left.\frac{\partial y_1}{\partial t}\right|_{t=0} = x + 1, \ \left.\frac{\partial y_2}{\partial t}\right|_{t=0} = x, \quad (42)$$

and at $t = T = 1$ we have the conditions:

$$y_1(1,x) = 2x^2 + x + 1,$$
$$y_2(0,x) = e^x + x + 1.$$
$$\left.\frac{\partial y_1}{\partial t}\right|_{t=1} = x + 1, \ \left.\frac{\partial y_2}{\partial t}\right|_{t=1} = x + 2. \quad (43)$$

From the system (41) we get the matrices B, D as follows:

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then, the system (7.1) become to:

$$\frac{\partial y_p(t,x)}{\partial t} = \frac{\partial y_p(t,x)}{\partial x},$$

we receive $p = 1, B_p = 1, D_p = 1 \ne 0$, $y_p = y_2(t,x), u_p = u_1(t,x)$. We will determine the conditions y_p from conditions from (42) to (43) as follows:

$$y_2(0,x) = e^x, \quad y_2(1,x) = e^x + x + 1. \quad (44)$$
$$\left.\frac{\partial y_2}{\partial t}\right|_{t=0} = x, \quad \left.\frac{\partial y_2}{\partial t}\right|_{t=1} = x + 2. \quad (45)$$
$$\left.\frac{\partial^2 y_2}{\partial t^2}\right|_{t=0} = \frac{\partial}{\partial x}\left.\left(\frac{\partial y_2}{\partial t}\right)\right|_{t=0} + \frac{\partial}{\partial x}\left.\left(\frac{\partial y_1}{\partial t}\right)\right|_{t=0} = 2 \quad (46)$$
$$\left.\frac{\partial^2 y_2}{\partial t^2}\right|_{t=1} = \frac{\partial}{\partial x}\left.\left(\frac{\partial y_2}{\partial t}\right)\right|_{t=1} + \frac{\partial}{\partial x}\left.\left(\frac{\partial y_1}{\partial t}\right)\right|_{t=1} = 2 \quad (47)$$

Then, $y_2(t,x)$ can be found as:

$$y_2(t,x) = \sum_{i=1}^{6} t^{i-1} c_i(x),$$

the coefficients $c_i(x)$ are determined according to the conditions $(44) - (47)$ as follows:

$$\begin{cases} c_1 = e^x \\ c_1 + c_2 + c_3 + c_4 + c_5 + c_6 = e^x + x + 1 \\ c_2 = x \\ c_2 + 2c_3 + 3c_4 + 4c_5 + 5c_6 = x + 2 \\ 2c_3 = 2 \\ 2c_3 + 6c_4 + 12c_5 + 20c_6 = 2. \end{cases}$$

Solve the above system, we get:

$$\begin{cases} c_1 = e^x \\ c_2 = x \\ c_3 = 1 \\ c_4 = c_5 = c_6 = 0. \end{cases}$$

Then we find $y_2(t,x)$ as the form:

$$y_2(t,x) = e^x + xt + t^2.$$

We have:

$$\frac{\partial y_2}{\partial t} = e^x + 2t.$$

Integrating the above expression with respect to x we get:

$$\int \frac{\partial y_2}{\partial t} dx = \int (e^x + 2t) dx = \frac{x^2}{2} + 2tx,$$

and

$$\int \frac{\partial y_2}{\partial x} dx = y_2(t, x) = e^x + xt + t^2.$$

Integrate both sides of equation (41) and based on the result just received above we have:

$$y_1(t, x) = \int \frac{\partial y_2}{\partial t} dx - y_2(t, x) = \frac{x^2}{2} - e^x + tx - t^2.$$

Putting the expression just received into the first equation of system (41) we yield:

$$u(t, x) = \frac{\partial y_1}{\partial t} = x - 2t.$$

8. Conclusion

This paper demonstrates that the solutions of the state function $y(t, x)$ and the control function $u(t, x)$ of the system $(1) - (5)$ can found in the polynomial form of degree $(2p + 4)$. The basis for the above problem is based on the fact that we can transform the descriptor control system (1) to the ceasing linear system (14), and then prove the possibility to define the degree of the system's solution in polynomial form. We then use the initial conditions $(2) - (5)$ to draw out the conclusion for $y(t, x)$ and $u(t, x)$. This study is expected to contribute to the field of control engineering by providing a more efficient and reliable method.

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REFERENCES

[1] L. H. Trung, *"Research on the solution structure of descriptor system"*. Project Science of Ministry of Education and Training, 2017, B2017. DNA. 09.

[2] S. P. Zubova, E. V. Raetskaya, and L.H. Trung, "On the solutions of the linear stationary control system", *Autom, Remote Control*, vol. 69, no. 11, pp. 1852-1858, 2008.

[3] S. P. Zubova and L.H. Trung "Construction of polynomail controls for linear stationary system with control points and additional constrains", *Autom, Remote Control*, vol. 71, no. 5, pp. 971-975, 2010.

[4] S. P. Zubova and E. V. Raetskaya, "Algorithm to solve linear multipoint proplems of control by the method of cascade decomposition", *Autom, Remote Control*, vol. 78, no. 7, pp. 1189-1202, 2017.

[5] S. P. Zubova and E. V. Raetskaya, "Construction of controls providing the desired output of the linear dynamic system", *Autom, Remote Control*, vol. 79, no. 5, pp. 774-779, 2017.

[6] L. H. Trung, H. T. T. Hai, and T. N. Tam, "Method cascade for control problem with multiple variables", *Quang Binh University – Journal of science & Technology*, vol. 21, no. 03, pp. 50-59, 2021.