PORT-HAMILTONIAN FORMULATION OF AN ELECTRICAL CIRCUIT USING DIFFERENT KINDS OF STATES

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Abstract - This work focuses on a dynamic electrical circuit whose dynamics are *affine* in the control input. Such dynamics are considered to be re-expressed in a canonical form, namely the port-Hamiltonian (pH) representation with dissipation, where the Hamiltonian is a quadratic function and has the unit of energy or power. On this basis, it allows revealing the transformation of energy (or power) inside the system, including the energy supply, storage and dissipation, thereby facilitating Lyapunov-based or energy-related control approaches for stabilization and optimization purposes. Two pH representations are proposed and compared; the first one is established with difficult-to-measure states while the second one is obtained with easier-to-measure states.

Key words - Electrical circuit; quadratic port-Hamiltonian representation; dissipation; states.

1. Introduction

This paper deals with dynamical systems [1-3] whose dynamics are described by a set of Ordinary Differential Equations (ODEs) and *affine* in the input u as follows:

$$\frac{dx}{dt} = f(x) + g(x)u; x(t=0) = x_{init} \tag{1}$$

where $x = x(t)$ is the state vector contained in the operating region $D \subset \mathbb{R}^n$, $f(x) \in \mathbb{R}^n$ expresses the smooth function with respect to the vector x. The input-state map and the control input are respectively represented by $g(x) \in \mathbb{R}^{n\times m}$ and $u \in \mathbb{R}^m$. Many industrial applications in, but not limited to, electrical, electromechanical, power and energy systems are governed by Eq. (1) [4-10].

For perspectives on passivity-based control, it is important to write the dynamics (1) into the port-Hamiltonian (pH) representation before developing state feedback laws for stabilization [11-19]. In other words, once a canonical form [20, 21], i.e. the pH formulation of the dynamics (1), is *a priori* derived, the interconnection and damping assignment passivity-based control (IDA-PBC) [11, 12], control by interconnection [13-15], energy/power shaping control [16, 17] and setpoint tracking control [18, 19], etc. can be advantageously applied to show stabilization properties. Hence, how to re-express the dynamics (1) in the pH representation is a key challenge in the port-based modeling research area, and it is the main subject of this work.

Notations: The following notations are considered throughout the paper:

- $\mathbb{R}$ is the set of real numbers.
- T is the matrix transpose.
- m and n $(m \le n)$ are positive integers.
- x_{init} is the initial value of the state vector.

2. An overview of port-Hamiltonian systems

In this section, we give a brief summary of pH systems, which can be used to re-express dynamical systems [20, 21] (the reader is also referred to [22], and references therein, for a preliminary description). Assume that the function $f(x)$ verifies the so-called separability condition [9, 23], that is, $f(x)$ can be decomposed and expressed as the product of some (interconnection and damping) structure matrices and the gradient of a potential function with respect to the state variables, i.e. the co-state variables:

$$f(x) = [\mathbf{J}(x) - \mathbf{R}(x)]\frac{\partial H(x)}{\partial x} \tag{2}$$

where $\mathbf{J}(x)$ and $\mathbf{R}(x)$ are the $n \times n$ skew-symmetric interconnection matrix (i.e. $\mathbf{J}(x) = -\mathbf{J}(x)^T$) and the $n \times n$ symmetric damping matrix (i.e. $\mathbf{R}(x) = \mathbf{R}(x)^T$), respectively, while $H(x): \mathbb{R}^n \to \mathbb{R}$ represents the Hamiltonian storage function of the system (possibly related to the total energy of the system). Furthermore, if the damping matrix $\mathbf{R}(x)$ is positive semi-definite, i.e.

$$\mathbf{R}(x) \ge 0, \tag{3}$$

then the dynamics (1) with (2) is a port-Hamiltonian (pH) representation with dissipation [20, 21]. On this basis, Eq. (1) is completed with the output and then rewritten as follows:

$$\begin{cases} \dfrac{dx}{dt} = [\mathbf{J}(x) - \mathbf{R}(x)]\dfrac{\partial H(x)}{\partial x} + g(x)u \\ y = g(x)^T\dfrac{\partial H(x)}{\partial x} \end{cases} \tag{4}$$

where y is the output.

It can be clearly seen for the pH model defined by Eqs. (3) and (4) that the time derivative of the Hamiltonian storage function $H(x)$ satisfies the energy balance equation (EBE) [16]

$$\frac{dH(x)}{dt} = \underbrace{-\left[\frac{\partial H(x)}{\partial x}\right]^T \mathbf{R}(x)\frac{\partial H(x)}{\partial x}}_{\text{dissipation}} + u^T y. \tag{5}$$

It can be shown from Eq. (3) that the dissipation term, defined by

$$d = -\left[\frac{\partial H(x)}{\partial x}\right]^T \mathbf{R}(x)\frac{\partial H(x)}{\partial x} \le 0 \tag{6}$$

is negative semi-definite. Hence, it represents a loss of energy due to resistive elements. The EBE (5) becomes:

$$\underbrace{\frac{dH(x)}{dt}}_{\text{stored power}} \leq \underbrace{u^T y}_{\text{supplied power}} \tag{7}$$

From a physical point of view, inequality (7) implies that the total amount of energy supplied from external source is always greater than the increase in the energy stored in the system. Hence, the pH system (4) is said to be *passive* with input u and output y corresponding to the Hamiltonian storage function $H(x)$ [2, 3]. This is one of advantageous features of the pH representation and has been applied for the control design. Indeed, under a zero - state detectability condition and the boundedness from below of the Hamiltonian storage function $H(x)$ by 0, it follows that an explicit proportional static output feedback law of the form [11, 23].

$$u = -K_p y \tag{8}$$

with $K_p > 0$ a so-called damping injection gain, renders the controlled pH system (4) dissipative and therefore asymptotically stabilized at the origin because $H(x)$ qualifies as a control Lyapunov function (we also refer the reader to [19] for further discussion).

In what follows, a series *RLC* circuit is used to illustrate and show the way to achieve a pH representation from given dynamics. For that purpose, the following lemma is adopted.

Lemma 1. Given the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It follows that:

i) $\mathbf{A} = \underbrace{\frac{\mathbf{A}-\mathbf{A}^T}{2}}_{\text{skew-symmetric}} + \underbrace{\frac{\mathbf{A}+\mathbf{A}^T}{2}}_{\text{symmetric}}$;

ii) if $\det(\mathbf{A}) \neq 0$ then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

3. On the pH formulations of a dynamic electrical circuit

3.1. Circuit description

To illustrate the concepts introduced in Section 2, we consider next a simple electrical system, which is the series *RLC* circuit as sketched in Figure 1.

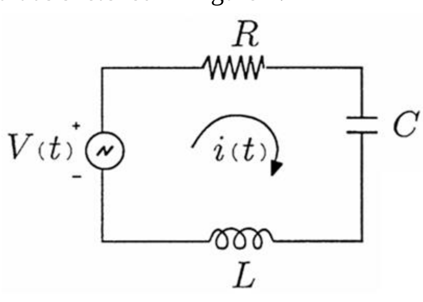

Figure 1. *A series RLC circuit [24]*

Before proceeding further, we remind Kirchhoff's voltage law:

$$u_L + u_R + u_C = V \tag{9}$$

and constitutive equations considered for three passive elements

$$\begin{cases} \text{the resistor } R\text{: } u_R = Ri_R \\ \text{the inductor } L\text{: } \emptyset_L = Li_L \text{ and } u_L = \frac{d\emptyset_L}{dt} \\ \text{the capacitor } C\text{: } i_C = \frac{dq_C}{dt} \text{ and } q_C = Cu_C \end{cases} \tag{10}$$

where q_C and $\emptyset_L$ are the electric charge stored in the capacitor C and the magnetic flux through the inductor L, respectively; while i is the electric current passing through the circuit ($i = i_R = i_C = i_L$) and u_L is the voltage of the inductor L (similarly for u_R and u_C).

3.2. Port-Hamiltonian formulation with difficult-to-measure states

Let $x := (q_C, \emptyset_L)^T$ be the vector consisting of the charge q_C and the magnetic flux through the conductor, that is $\emptyset_L = L\frac{dq_C}{dt}$. From Eqs. (9) and (10), one has [16, 24]:

$$\frac{dq_C}{dt} = \frac{1}{L}\emptyset_L\,, \tag{11}$$

$$\frac{d\emptyset_L}{dt} = -\frac{1}{C}q_C - \frac{R}{L}\emptyset_L + V. \tag{12}$$

The following proposition summarizes the related results published in [24].

Proposition 1. Equations (11) and (12) correspond to a pH representation described by (4) with $x := (q_C, \emptyset_L)^T$ and

$$\mathbf{J}(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{13}$$

$$\mathbf{R}(x) = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}, \tag{14}$$

$$g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{15}$$

$$u = V, \tag{16}$$

$$y = \frac{1}{L}\emptyset_L. \tag{17}$$

Furthermore, the system is passive with the Hamiltonian defined by

$$H(x) = \frac{1}{2C}q_C^2 + \frac{1}{2L}\emptyset_L^2. \tag{18}$$

Proof. It follows from Eqs. (1), (11) and (12) that $f(x) = \begin{pmatrix} \frac{1}{L}\emptyset_L \\ -\frac{1}{C}q_C - \frac{R}{L}\emptyset_L \end{pmatrix}$, which can be rewritten as $f(x) = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -R \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \frac{1}{C}q_C \\ \frac{1}{L}\emptyset_L \end{pmatrix} \equiv [\mathbf{J}(x) - \mathbf{R}(x)]\frac{\partial H(x)}{\partial x}$ (using Lemma 1). This concludes the proof. ∎

Remark 1. The Hamiltonian (18) is equal to the total energy of the system (i.e., it characterizes the amount of energy stored in capacitor and inductor). Hence it has the unit of energy [24]. Consequently, the dissipation term is strongly and explicitly related to the resistor of the circuit. Indeed, it can be shown from Eq. (6) that

$$d_{\text{PROP1}} = -\frac{1}{L}\emptyset_L R \frac{1}{L}\emptyset_L = -R\left(\frac{1}{L}\emptyset_L\right)^2 \leq 0. \tag{19}$$

Remark 2. From a practical point of view, the vector x contains the states $q_C(t)$ and $\emptyset_L(t)$ which are difficult to measure due to the lack of appropriate devices or the cost and feasibility of installing sensors. Thus, the design of a state-feedback controller may be worse whenever those states are unavailable.

3.3. Port-Hamiltonian formulation with easier-to-measure states

From Eqs. (9) and (10), it is possible to write:

$$\frac{du_C}{dt} = \frac{1}{C}\frac{dq_C}{dt} = \frac{1}{C}i = \frac{1}{C}i_L\,, \tag{20}$$

$$\frac{di_L}{dt} = -\frac{R}{L}i_L - \frac{1}{L}u_C + \frac{1}{L}V. \tag{21}$$

Now, let x denote the vector consisting of the capacitor voltage and inductor current, i.e. $x \coloneqq (u_C, i_L)^T$. We state the following proposition, which highlights the novelty of this work, compared to the previous one [24].

Proposition 2. Equations (20) and (21) correspond to a pH representation described by (4) with $x \coloneqq (u_C, i_L)^T$,

$$\mathbf{J}(x) = \begin{pmatrix} 0 & \frac{R}{L} \\ -\frac{R}{L} & 0 \end{pmatrix}, \tag{22}$$

$$\mathbf{R}(x) = \begin{pmatrix} 0 & 0 \\ 0 & C\left(\frac{R}{L}\right)^2 \end{pmatrix}, \tag{23}$$

$$g(x) = \frac{1}{L}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{24}$$

$$y = \tfrac{1}{RC}i_L, \tag{25}$$

and u given in Eq. (16).

Furthermore, the system is passive with the Hamiltonian defined by

$$H(x) = \frac{1}{2R}u_C^2 + \frac{L}{2RC}i_L^2. \tag{26}$$

Proof. It follows from Eqs. (1), (20) and (21) that $f(x) = \begin{pmatrix} \frac{1}{C}i_L \\ -\frac{1}{L}u_C - \frac{R}{L}i_L \end{pmatrix}$. By multiplying both sides of Eq. (1) with $\mathbf{Q}(x) = \begin{pmatrix} -C & -\frac{L}{R} \\ \frac{L}{R} & 0 \end{pmatrix}$, one has

$$\mathbf{Q}(x)\frac{dx}{dt} = \mathbf{Q}(x)f(x) + \mathbf{Q}(x)g(x)u,$$

where $\mathbf{Q}(x)f(x) = \begin{pmatrix} \frac{1}{R}u_C \\ \frac{L}{RC}i_L \end{pmatrix}$. As the matrix $\mathbf{Q}(x)$ is invertible, one obtains

$$\frac{dx}{dt} = \mathbf{Q}^{-1}(x)\begin{pmatrix} \frac{1}{R}u_C \\ \frac{L}{RC}i_L \end{pmatrix} + g(x)u$$

with $\mathbf{Q}^{-1}(x) = \begin{pmatrix} 0 & \frac{R}{L} \\ -\frac{R}{L} & -C\left(\frac{R}{L}\right)^2 \end{pmatrix} \equiv \mathbf{J}(x) - \mathbf{R}(x)$ (using Lemma 1). This completes the proof. ■

Remark 3. The existence of the matrix $\mathbf{Q}(x)$ is not unique, that is, any matrix of the form $\lambda\mathbf{Q}(x), (\lambda \in \mathbb{R}_+^*)$ is also qualified for the formulation. Note that $\mathbf{Q}(x)$ can also be used in obtaining the Brayton-Moser form of the system [17].

Remark 4. It can be checked that the Hamiltonian (26) has the unit of power (note that RC characterizes the time constant of the system). Consequently, the dissipation term is no longer related to the resistor R of the circuit for the obtained pH formulation. Indeed, it can be shown from Eq. (6) that

$$d_{\text{PROP2}} = -\left(\frac{L}{RC}i_L\right)^2 C\left(\frac{R}{L}\right)^2 = -\frac{1}{C}i_L^2 \le 0. \tag{27}$$

Remark 5. The vector x contains the states $u_C(t)$ and $i_L(t)$ which are easy to measure (for example, using an oscilloscope or a multimeter). Thus, the design of a state-feedback controller in this case is of practical interest. On the other hand, adopting these easy-to-measure states is well suited for the application of machine learning for pH realizations in terms of modeling and learning-based control [25]. Indeed, as the resulting pH system is linear, $H(x)$ (Eq. (26)) can be rewritten as

$$H(x) = \frac{1}{2}x^T\mathbf{M}x\,, \tag{28}$$

with $\mathbf{M} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{L}{RC} \end{pmatrix}$. The Williamson decomposition reads

$$\mathbf{M} = \mathbf{S}^T\mathbf{D}\mathbf{S}\,, \tag{29}$$

with $\mathbf{S} = \begin{pmatrix} \left(\frac{L}{C}\right)^{-1/4} & 0 \\ 0 & \left(\frac{L}{C}\right)^{1/4} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} \frac{1}{R}\left(\frac{L}{C}\right)^{1/2} & 0 \\ 0 & \frac{1}{R}\left(\frac{L}{C}\right)^{1/2} \end{pmatrix}$. Thus, the key idea is to learn the parameters defining the pH system associated with $(\mathbf{S}, \mathbf{D})$.

Remark 6. If the matrix $\mathbf{Q}(x)$ is chosen as $\mathbf{Q}(x) = \begin{pmatrix} -RC^2 & -LC \\ LC & 0 \end{pmatrix}$, i.e. $\mathbf{Q}^{-1}(x) = \begin{pmatrix} 0 & \frac{1}{LC} \\ -\frac{1}{LC} & -R\frac{1}{L^2} \end{pmatrix}$, then the results in Proposition 2 become

$$\mathbf{J}(x) = \begin{pmatrix} 0 & \frac{1}{LC} \\ -\frac{1}{LC} & 0 \end{pmatrix}, \tag{30}$$

$$\mathbf{R}(x) = \begin{pmatrix} 0 & 0 \\ 0 & R\frac{1}{L^2} \end{pmatrix}, \tag{31}$$

$$y = i_L, \tag{32}$$

while u and $g(x)$ remain unchanged as per Eqs. (16) and (24), respectively. In addition, the system is passive with the Hamiltonian defined by

$$H(x) = \frac{C}{2}u_C^2 + \frac{L}{2}i_L^2. \tag{33}$$

Note that y given in Eq. (32) is precisely the output defined in Eq. (17), while the Hamiltonian (33) can be re-expressed in terms of q_C and $\emptyset_L$ as per Eq. (18) due to Eq. (10), i.e. it still has the unit of energy.

Table 1 summarizes the main features of the two proposed pH formulations.

Table 1. *Comparison of the two pH models*

	The pH model with difficult-to-measure states	The pH model with easier-to-measure states
x	$(q_C, \emptyset_L)^T$ (difficult to measure)	$(u_C, i_L)^T$ (easier-to-measure)
$\mathbf{J}(x)$	given by Eq. (13)	given by Eq. (22)
$\mathbf{R}(x)$	given by Eq. (14)	given by Eq. (23)
$g(x)$	given by Eq. (15)	given by Eq. (24)
u	V	V
y	given by Eq. (17)	given by Eq. (25)
$H(x)$	given by Eq. (18) (unit of energy)	given by Eq. (26) (unit of power)

The advantages of the pH formulation with easier-to-measure states can be summarized as follows; (i) its pH (state-space) model is based on states which are easy to measure, thereby supporting the design of state-feedback controllers for stabilization, and (ii) the Hamiltonian remains a different physical interpretation, that is, it has the unit of power.

4. Conclusion

In this paper, the pH formulations of a transient series *RLC* circuit are proposed and compared using different kinds of states, namely the difficult- or easier-to-measure states. The resulting Hamiltonian is a quadratic function, which has the unit of either energy or power.

It remains now to extend the approach to power electronic circuits, and adapt the power-shaping control [17], setpoint tracking control theory [19] or learning-based control [25] to stabilize the systems at a desired setpoint.

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