LINEAR SYSTEM OF CUBIC FORMS WITH BASE LOCUS OF 5 POINTS

HỆ TUYẾN TÍNH CỦA CÁC DẠNG BẬC 3 VỚI 5 ĐIỂM CƠ SỞ

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Abstract - Studying a variety through a linear system of homogenous forms with a base locus is a method that has many advantages not only for understanding important geometric properties but also the ability to accurately evaluate many invariants of the variety. This paper aims to study linear systems of cubic forms through 5 points in the projective plane with many configurations. Many interesting and profound properties of surfaces have been discovered through their corresponding linear systems. These results not only enrich correspondence theory in algebraic geometry and computational algebra but also promise to provide methods and tools for applications in the fields of information technology and other technologies.

Key words - Varieties and morphism; computational algebra; singularity; linear system; blowing-up

Tóm tắt – Nghiên cứu một đa tạp thông qua hệ tuyến tính của các dạng thuần nhất với một cơ sở cho trước là một phương pháp có nhiều ưu điểm không những để hiểu rõ các tính chất hình học mà còn có thể cho phép tính toán chính xác nhiều bất biến của đa tạp. Bài báo này nghiên cứu hệ tuyến tính của các dạng bậc ba với cơ sở gồm 5 điểm có nhiều cấu hình khác nhau trong mặt phẳng xạ ảnh. Nhiều kết quả thú vị và sâu sắc của đa tạp nhận được thông qua nghiên cứu tính chất của hệ tuyến tính tương ứng của nó. Các kết quả này không những làm phong phú thêm cho các lĩnh vực lý thuyết của hình học đại số và đại số tính toán mà còn cung cấp các công cụ ứng dụng trong công nghệ thông tin và các lĩnh vực công nghệ khác.

Từ khóa – Đa tạp và cấu xạ; đại số tính toán; kỳ dị; hệ tuyến tính; phép nổ

1. Introduction

Let $\mathcal{P} = \{P_1, P_2, ..., P_r\}$ be a set of r distinct points in the projective plane $\mathbb{P}^2$. Denote $\mathcal{L}_\mathcal{P}$ the linear system of homogenous forms of same degree through $\mathcal{P}$. Let Supp($\mathcal{L}_\mathcal{P}$) denote the variety defined by $\mathcal{L}_\mathcal{P}$. Suppose that dim($\mathcal{L}_\mathcal{P}$) $=m$, let $\{F_1, F_2, ..., F_m\}$ be a basis of $\mathcal{L}_\mathcal{P}$. Consider the morphism:

$$\varphi: \mathbb{P}^2 - \text{Supp}(\mathcal{L}_\mathcal{P}) \to \mathbb{P}^{m-1}$$
$$P \mapsto \big(F_1(P): ... : F_m(P)\big). \quad (1)$$

Let $X = \overline{Im(\varphi)}$.

Lemma 1.1. The variety X defined in (1) does not depend on the choice of basis and if φ is injective in an open subset, then X is a surface with degree equal to the number of intersection points of two generic curves in $\mathcal{L}_\mathcal{P}$ outside of Supp($\mathcal{L}_\mathcal{P}$).

Proof. Let $\{G_1, G_2, ..., G_r\}$ be another basis of $\mathcal{L}_\mathcal{P}$, and let X' be the variety defined in (1) with respect to the basis. Let $M \in GL(m)$ be the matrix of base change from $\{F_1, F_2, ..., F_r\}$ to $\{G_1, G_2, ..., G_r\}$. Then M defines a transformation of $\mathbb{P}^{m-1}$ which map X to X'. Let C_1, C_2 be two generic forms in $\mathcal{L}_\mathcal{P}$ with a given basis E. The coordinates of C_1, C_2 in E correspond to 2 hyperplanes H_1, H_2 in $\mathbb{P}^{m-1}$. The intersection of $H_1 \cap H_2$ with X equals to number of $C_1 \cap C_2$ outside of $\mathcal{L}_\mathcal{P}$.

Remark 1.1. From now on, the variety X as in (1) is called the variety defined by $\mathcal{L}_\mathcal{P}$.

Studying variety X through a linear system $\mathcal{L}_\mathcal{P}$ is a method that has many advantages not only for understanding important geometric properties as irreducibility, singularity, completed intersection, but also the ability to accurately evaluate dimension and degree of X, the number of singularities and how they are obtained, the number of lines and how they are obtained, and other quantitative invariants of X, see [5, 6].

It is well-known that the linear system of cubic forms containing 6 points in general position corresponds to a smooth cubic surface with 27 lines in $\mathbb{P}^3$, see [1, 2, 3, 4, 7, 8]. In recent years, further research on linear systems of cubic forms through 6 points not in general position has shown interesting results corresponding to singular, semi-stable cubic surfaces ([8, 10]). Not only the properties of singularity and the number of straight lines on cubic surfaces can be recognized through the corresponding linear system, but specializations of 6-point configurations also reveals profound properties on the boundaries of moduli space of non-singular cubic surfaces with Eckardt points ([9, 11]).

This paper aims to study projective varieties corresponding to a linear system of cubic forms through 5 points in $\mathbb{P}^2$. Varieties X in various cases of configurations of 5 points are surfaces in $\mathbb{P}^4$ with degree 3 or 4. Many interesting and profound properties of surfaces X have been discovered through their corresponding linear systems.

2. Linear system of cubic forms with base locus of 5 points

Theorem 2.1. Let $\mathcal{P} = \{P_1, ..., P_5\}$ be a set of 5 points in general position in $\mathbb{P}^2$. Let $\mathcal{L}_\mathcal{P}$ be the linear system of cubic forms through $\mathcal{P}$. Then the variety X defined by $\mathcal{L}_{\mathcal{P}}$ is a smooth surface of degree 4 in $\mathbb{P}^4$ with 10 lines.

Proof. It is clear that the Supp($\mathcal{L}_{\mathcal{P}}$) = $\mathcal{P}$, the vector space $\mathcal{L}_{\mathcal{P}}$ has dimension 5 and the map φ as in (1) is injective. Two generic cubics C_1, C_2 have 4 points in common outside $\mathcal{P}$. It means that X is a surface of degree 4. Moreover, the line $l_{ij} = \overline{P_iP_j}$ with $1 \le i < j \le 5$, intersects any cubic curve in $\mathcal{L}_{\mathcal{P}}$ at another point. It implies that the image of l_{ij} is a line on surface X.

We can assume that P_1, P_2, P_3, P_4 are projective system points of $\mathbb{P}^2$ and $P_5 = (a: b: 1)$ where $a, b \in k$ such that $\mathcal{P}$ are in general position. A direct computation shows that with a specific basis of $\mathcal{L}_{\mathcal{P}}$ (as in ***Remark 2.1***), the surface $X = V(g_1, g_2) \subset \mathbb{P}^4$, where

$$g_1 = -xt + \frac{(a^2-b)y^2}{b-a} + yz - \frac{b(a-1)yt}{b-a} - \frac{a(a-1)ys}{b-a} + \frac{a(b-1)ts}{b-a},$$

$$g_2 = -xy + \frac{(b^2-a)y^2}{b-a} - \frac{b(b-1)yt}{b-a} - \frac{a(b-1)ys}{b-a} + zs + \frac{b(a-1)ts}{b-a}$$

and X is smooth.

Remark 2.1. Suppose that $P_1 = (1: 0: 0)$, $P_2 = (0: 1: 0)$, $P_3 = (0: 0: 1)$, $P_4 = (1: 1: 1)$ and $P_5 = (a: b: 1)$ such that $\mathcal{P}$ is in general position. A specific basis of $\mathcal{L}_{\mathcal{P}}$ could be

$$f_1 = uv^2 + \frac{b(ab-1)uw^2}{b-a} - \frac{a(b^2-1)vw^2}{b-a},$$

$$f_2 = uvv + \frac{b(a-1)uw^2}{b-a} - \frac{a(b-1)vw^2}{b-a},$$

$$f_3 = u^2v + \frac{b(a^2-1)uw^2}{b-a} - \frac{a(ab-1)vw^2}{b-a},$$

$$f_4 = u^2w + \frac{(a^2-b)uw^2}{b-a} - \frac{a(a-1)vw^2}{b-a},$$

$$f_5 = v^2w + \frac{b(b-1)uw^2}{b-a} - \frac{(b^2-a)vw^2}{b-a}.$$

Theorem 2.2. Let $\mathcal{P} = \{P_1, ..., P_5\} \subset \mathbb{P}^2$ not in general position and have 4 points in general position. Let $\mathcal{L}_{\mathcal{P}}$ be the linear system of cubic forms through $\mathcal{P}$. Then the variety X defined by $\mathcal{L}_{\mathcal{P}}$ is a smooth surface of degree 4 in $\mathbb{P}^4$ with at most 7 lines.

Proof. Since 5 points in $\mathcal{P}$ are not in general position, we can assume that P_1, P_2, P_5 are on a line l and P_1, P_2, P_3, P_4 are in general position. It is clear that the Supp($\mathcal{L}_{\mathcal{P}}$) = $\mathcal{P}$. The vector space $\mathcal{L}_{\mathcal{P}}$ has dimension 5 and the map φ as in (1) is injective outside l. The image of l is one point on X. Two generic cubics C_1, C_2 have 4 points in common outside $\mathcal{P}$. It means that X is a surface of degree 4. Moreover, each line in $\overline{P_4P_j}, \overline{P_3P_j}$ with j = 1, 2, 5 intersects any cubic curve in $\mathcal{L}_{\mathcal{P}}$ at one point outside $\mathcal{P}$. It implies that the images of $\overline{P_4P_j}, \overline{P_3P_j}$ are 6 lines on surface X. If the line $\overline{P_3P_4}$ does not contain any of P_1, P_2, P_5, then its image is another line on X.

We can assume that $P_1 = (1: 0: 0)$, $P_2 = (0: 1: 0)$, $P_3 = (0: 0: 1)$, $P_4 = (1: 1: 1)$ are projective system points of $\mathbb{P}^2$ and $P_5 = (a: b: 0)$. A direct computation shows that with a specific basis of $\mathcal{L}_{\mathcal{P}}$ (as in ***Remark 2.2***), the surface $X = V(g_1, g_2) \subset \mathbb{P}^4$, where

$$g_1 = xz - xs - \frac{by^2}{a} + \frac{(b-a)yz}{a} + yt,$$

$$g_2 = y^2 - yz - ys + zt$$

and X is smooth.

Remark 2.2. A specific basis of $\mathcal{L}_{\mathcal{P}}$ could be

$$f_1 = uv^2 - \frac{bu^2v}{a} + \frac{(b-a)u^2w}{a},\ f_2 = -u^2w + uvw,$$

$$f_3 = -u^2w + uw^2,\ f_4 = -u^2w + v^2w,\ f_5 = -u^2w + vw^2.$$

The image of line $l = V(w)$ is (1: 0: 0: 0: 0) in $\mathbb{P}^4$.

Theorem 2.3. Let $\mathcal{P} = \{P_1, ..., P_5\} \subset \mathbb{P}^2$ with 4 points in general position. Suppose that P_1, P_2, P_5 are colinear, the points P_3, P_4, P_5 are colinear. Let $\mathcal{L}_{\mathcal{P}}$ be the linear system of cubic forms through $\mathcal{P}$. Then the variety X defined by $\mathcal{L}_{\mathcal{P}}$ is a smooth surface of degree 4 in $\mathbb{P}^4$ with 4 lines.

Proof. We can assume that P_1, P_2, P_3, P_4 are in general position. Let l_1 be the line containing P_1, P_2, P_5, let l_2 be the line containing P_3, P_4, P_5. It is clear that the Supp($\mathcal{L}_{\mathcal{P}}$) = $\mathcal{P}$. The vector space $\mathcal{L}_{\mathcal{P}}$ has dimension 5 and the map φ as in (1) is injective outside $l_1 \cup l_2$. The image of l_i for i = 1, 2 is one point on X. Two generic cubics C_1, C_2 have 4 points in common outside $\mathcal{P}$. It means that X is a surface of degree 4. Moreover, each line in $\overline{P_1P_3}, \overline{P_1P_4}, \overline{P_2P_3}, \overline{P_2P_4}$ intersects any cubic curve in $\mathcal{L}_{\mathcal{P}}$ at one point outside of $\mathcal{P}$. This implies that the images of them are 4 lines on surface X.

We can assume that $P_1 = (1: 0: 0)$, $P_2 = (0: 1: 0)$, $P_3 = (0: 0: 1)$, $P_4 = (1: 1: 1)$ are projective system points of $\mathbb{P}^2$ and $P_5 = (1: 1: 0)$. A direct computation shows that with a specific basis of $\mathcal{L}_{\mathcal{P}}$ (as in ***Remark 2.3***), the surface $X = V(g_1, g_2) \subset \mathbb{P}^4$ where

$$g_1 = xz - xs - y^2 + yt,$$

$$g_2 = y^2 - yz - ys + zt$$

and X is smooth.

Remark 2.3. A specific basis of $\mathcal{L}_{\mathcal{P}}$ could be

$$f_1 = uv^2 - u^2v, \qquad f_2 = -u^2w + uvw,$$
$$f_3 = -u^2w + uw^2, \qquad f_4 = -u^2w + v^2w,$$
$$f_5 = -u^2w + vw^2.$$

The image of line $l_1 = V(w)$ is (1:0:0:0:0) and the image of line $l_2 = V(u-v)$ is (0:0:1:0:1).

Theorem 2.4. Let $\mathcal{P} = \{P_1, ..., P_5\} \subset \mathbb{P}^2$ with exactly 4 points on a line. Let $\mathcal{L}_{\mathcal{P}}$ be the linear system of cubic forms through $\mathcal{P}$. Then the variety X defined by $\mathcal{L}_{\mathcal{P}}$ is a surface of degree 3 in $\mathbb{P}^4$ with infinite number of lines.

Proof. We can assume that P_1, P_2, P_3, P_4 are on a line l. It is clear that the Supp($\mathcal{L}_{\mathcal{P}}$) = $l \cup \{P_5\}$ and each cubic form in $\mathcal{L}_{\mathcal{P}}$ factors into l and one quadratic containing P_5. The vector space $\mathcal{L}_{\mathcal{P}}$ has dimension 5 and the map φ as in (1) is injective outside $l \cup \{P_5\}$. Two generic cubics in $\mathcal{L}_{\mathcal{P}}$ have 3 common points outside Supp($\mathcal{L}_{\mathcal{P}}$). This means that the surface X has degree 3.

Any line containing P_5 intersects a generic cubic of $\mathcal{L}_{\mathcal{P}}$ at exactly one point outside Supp($\mathcal{L}_{\mathcal{P}}$). Then its image is a line on X. Therefore, X contains infinite number of lines.

3. Conclusion

The paper showed projective varieties corresponding to linear systems of cubic forms through 5 points in $\mathbb{P}^2$ with 4 different configurations. Varieties X corresponding to different configurations are surfaces in $\mathbb{P}^4$ with degree 3 or 4. Many interesting and profound properties of surfaces X have been discovered through their corresponding linear systems such as number of lines, degree, defining polynomials and others. These results not only enrich correspondence theory in algebraic geometry and computational algebra but also promise to provide methods and tools for applications in the fields of information technology and other technologies.

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