ω-COVER AND RELATED SPACES ON THE VIETORIS HYPERSPACE $\mathcal{F}(X)$

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Abstract - Recently, Tuyen et al. [1] showed that a space X has a σ-(P)-strong network consisting of cs-covers (resp., cs^*-covers) if and only if the hyperspace $\mathcal{F}(X)$ does, where (P) is one of the following properties: point finite, point countable, compact finite, compact-countable, locally finite, locally countable. Moreover, the author [1] also proved that X is a Cauchy sn-symmetric space with a σ-(P)-property cs^*-network (resp., cs-network, sn-network.) if and only if so is $\mathcal{F}(X)$. In this paper, we study the concepts of ω-cover and certain spaces defined by ω-covers on the hyperspace $\mathcal{F}(X)$ of finite subsets of a space X endowed with the Vietoris topology. We prove that X is an ω-Lindelöf (resp., ω-Menger, ω-Rothberger) space if and only if so is $\mathcal{F}(X)$.

Key words - ω-cover; Vietoris hyperspace; ω-Lindelöf; ω-Menger; ω-Rothberger.

1. Introduction

The study of generalized metric properties on hyperspaces with Vietoris topology has been a significant focus of research (see [2-9]). In 2021, F. Lin, R. Shen, and C. Liu investigated the heredity of the classes of generalized metric spaces (such as Nagata spaces, c-semi-stratifiable spaces, γ-spaces, and semi-metrizable spaces) to hyperspace of nonempty finite subsets with the Vietoris topology (see [10]). Most recently, L.Q. Tuyen, O.V. Tuyen, and Lj.D.R. Kočinac showed that a space X has a σ-(P)-strong network consisting of cs-covers (cs^*-covers) if and only if the hyperspace $\mathcal{F}(X)$ does, where (P) is one of the following properties: point-finite, point-countable, compact-finite, compact-countable, locally finite, locally countable (see [1]). In this paper, we study the concepts of ω-cover and certain spaces defined by ω-covers such as ω-Lindelöf, ω-Menger, ω-Rothberger on the Vietoris hyperspace $\mathcal{F}(X)$.

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, all spaces are assumed to be Hausdorff, other concepts and terms are understood in their usual sense unless otherwise specified (see [11]). Moreover, if $\mathcal{A}$ is a family of subsets of a topology space X, then

$$\bigcup\mathcal{A} = \bigcup\{A : A \in \mathcal{A}\}.$$

2. Theoretical Background and Research Methods

2.1. Theoretical Background

Let X be a space, then its *hyperspaces be defined resp. by*

(1) $CL(X) = \{A \subset X : A \text{ is closed and nonempty}\}$;

(2) $\mathbb{K}(X) = \{A \in CL(X) : A \text{ is compact}\}$;

(3) $\mathcal{F}_n(X) = \{A \in CL(X) : |A| \le n\}$, where $n \in \mathbb{N}$;

(4) $\mathcal{F}(X) = \{A \in CL(X) : A \text{ is finite}\}$.

The set $CL(X)$ is topologized by the *Vietoris topology* defined as the topology generated by

$$\mathcal{B} = \left\{\langle U_1,\dots,U_k\rangle : U_1,\dots,U_k \text{ are open subsets of } X, k \in \mathbb{N}\right\},$$

where

$$\langle U_1,\dots,U_k\rangle = \left\{A \in CL(X) : A \subset \bigcup_{i\le k} U_i,\ A \cap U_i \ne \varnothing\ \forall i \le k\right\}.$$

Note that, by definition, $\mathbb{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subspaces of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

(1) $CL(X)$ is called the hyperspace of nonempty closed subsets of X;

(2) $\mathbb{K}(X)$ is called the *hyperspace of nonempty compact subsets of* X;

(3) $\mathcal{F}_n(X)$ is called the n*-fold symmetric product of* X;

(4) $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of* X.

On the other hand, it is obvious that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty}\mathcal{F}_n(X)$ and $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in \mathbb{N}$.

If $U_1,\dots,U_s$ are open subsets of a topology space (X,τ), then $\langle U_1,\dots,U_s\rangle_{\mathcal{F}(X)}$ denotes the intersection of the open set $\langle U_1,\dots,U_s\rangle$ of the Vietoris topology with $\mathcal{F}(X)$.

Remark 2.1.1 ([7]). Let X be a space and let $n \in \mathbb{N}$.

(1) $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$.

(2) $f_1 : X \twoheadrightarrow \mathcal{F}_1(X)$ given by $f_1(x) = \{x\}$ is a homeomorphism.

(3) Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m, n \in \mathbb{N}$, $m < n$.

Definition 2.1.2 ([12])**.** Let X be a space and $\mathcal{U}$ be an open cover of X. Then, $\mathcal{U}$ is said to be an ω*-cover* of X if every finite subset of X is contained in a member of $\mathcal{U}$.

Definition 2.1.3 ([12])**.** Let X be a space. Then, X is said to be

(1) ω*-Lindelöf* if every ω-cover of X has a countable subset which is an ω-cover of X.

(2) ω*-Menger* if for every $\{\mathcal{U}_n : n\in\mathbb{N}\}$ is a sequence of ω-covers of X, there exists a sequence $\{\mathcal{V}_n : n\in\mathbb{N}\}$ such that for each $n\in\mathbb{N}$, $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\mathcal{V}_n$ is an ω-cover of X.

(3) ω*-Rothberger* if for every $\{\mathcal{U}_n : n\in\mathbb{N}\}$ is a sequence of ω-covers of X, there exists, for each $n\in\mathbb{N}$, $U_n\in\mathcal{U}_n$ such that $\{U_n : n\in\mathbb{N}\}$ is an ω-cover of X.

2.2. Research Methods

In the research process, we collected and analyzed papers from previous researchers. Then, by applying methods of analogy and generalization to some results from these works, the authors established the main results for their paper.

3. Research Results and Discussion

3.1. Research Results

Let A be a closed subspace of a space X and $\mathcal{U}_A$ be an open cover of A. Then, for each $U\in\mathcal{U}_A$, there exists V_U is an open subset of X such that

$$U = V_U \cap A.$$

If we put

$$\mathcal{U} = \{V_U \cup (X\setminus A) : U\in\mathcal{U}_A\},$$

then, $\mathcal{U}$ is an open cover of X.

Proof. Because A is a closed subspace of a space X, $X\setminus A$ is an open subset of X. Then, $V_U\cup(X\setminus A)$ is an open subset of X for every $U\in\mathcal{U}_A$. Now, we prove that

$$\mathcal{U} = \{V_U \cup (X\setminus A) : U\in\mathcal{U}_A\}$$

is an open cover of X. Indeed, since

$$X = A\cup(X\setminus A) = \left(\bigcup_{U\in\mathcal{U}_A} U\right)\cup(X\setminus A)$$

$$= \bigcup_{U\in\mathcal{U}_A}\big(U\cup(X\setminus A)\big)\subset\bigcup_{U\in\mathcal{U}_A}\big(V_U\cup(X\setminus A)\big),$$

$\mathcal{U}$ is an open cover of X.

Lemma 3.1.1. Let A be a closed subspace of a space X. If $\mathcal{U}_A$ *is an* ω-cover of A, then $\mathcal{U}$ is ω-open cover of X.

Proof. Suppose that $\mathcal{U}_A$ is an ω-cover of A and F is a finite subset of X. Then, $F\cap A$ is a finite subset of A. Hence, there exists $U_F\in\mathcal{U}_A$ such that

$$F\cap A\subset U_F.$$

This implies that

$$F = (F\cap A)\cup(F\setminus A)\subset U_F\cup(X\setminus A)\subset V_{U_F}\cup(X\setminus A).$$

Thus, $\mathcal{U}$ is an ω-cover of X.

Lemma 3.1.2. *Let A be a closed subspace of a space X. If X is an ω-Lindelöf space, then so is A.*

Proof. Suppose that $\mathcal{U}_A$ is an ω-cover of A. By Lemma 3.1.1, $\mathcal{U}$ is an ω-cover of X. Because X is an ω-Lindelöf space, there exists a sequence

$$\{U_n : n\in\mathbb{N}\}\subset\mathcal{U}_A$$

such that

$$\{V_{U_n}\cup(X\setminus A) : n\in\mathbb{N}\}$$

is an ω-cover of X. Now, we will prove that

$$\{U_n : n\in\mathbb{N}\}$$

is an ω-cover of A. Indeed, let F be a finite set in A. Then, F is a finite set in X. This implies that there exists $n_F\in\mathbb{N}$ such that

$$F\subset V_{U_{n_F}}\cup(X\setminus A).$$

Since F is a subset of A, so F is a subset of $V_{U_{n_F}}$. It shows that

$$F\subset V_{U_{n_F}}\cap A = U_{n_F}.$$

Therefore, $\{U_n : n\in\mathbb{N}\}$ is an ω-cover of A. Hence, A is an ω-Lindelöf space.

Lemma 3.1.3. *Let A be a closed subspace of a space X. If X is an ω-Menger (resp., ω-Rothberger) space, then so is A.*

Proof. Assume that $\{\mathcal{U}_n : n\in\mathbb{N}\}$ is a sequence of ω-covers of A. For each $n\in\mathbb{N}$, we put

$$\mathcal{V}_n = \{V_U\cup(X\setminus A) : U\in\mathcal{U}_n\}.$$

By Lemma 3.1.1, $\{\mathcal{V}_n : n\in\mathbb{N}\}$ is a sequence of ω-covers of X.

Case 1. If X is an ω-Menger space, then for each $n\in\mathbb{N}$, there exists $\mathcal{W}_n$ is a finite subfamily of $\mathcal{U}_n$ such that

$$\bigcup_{n\in\mathbb{N}}\{V_U\cup(X\setminus A) : U\in\mathcal{W}_n\}$$

is an ω-cover of X. Now, we will prove that $\bigcup_{n\in\mathbb{N}}\mathcal{W}_n$ is an ω-cover of A. Indeed, let F be a finite subset of A. Then, F is a finite subset of X. This implies that there exist $n_F\in\mathbb{N}$ and $W_{n_F}\in\mathcal{W}_{n_F}$ such that

$$F\subset V_{W_{n_F}}\cup(X\setminus A).$$

Since F is a subset of A, so F is a subset of $V_{W_{n_F}}$, this implies that

$$F\subset V_{W_{n_F}}\cap A = W_{n_F}.$$

Therefore, $\bigcup_{n\in\mathbb{N}}\mathcal{W}_n$ is an ω-cover of A. Hence, A is an ω-Menger space.

Case 2. If X is an ω-Rothberger space, then for each $n\in\mathbb{N}$, there exists $U_n\in\mathcal{U}_n$ such that

$$\left\{V_{U_n}\cup(X\setminus A):n\in\mathbb{N}\right\}$$

is an ω-cover of X. Now, we will prove that

$$\left\{U_n:n\in\mathbb{N}\right\}$$

is an ω-cover of A. Indeed, let F be a finite subset of A. Then, F is a finite subset of X. This implies that there exists $n_F\in\mathbb{N}$ such that

$$F\subset V_{U_{n_F}}\cup(X\setminus A).$$

Since F is a subset of A, so F is a subset of $V_{U_{n_F}}$. This shows that $F\subset V_{U_{n_F}}\cap A=U_{n_F}$.

Therefore, $\left\{U_n:n\in\mathbb{N}\right\}$ is an ω-cover of A. Hence, A is an ω-Rothberger space.

Lemma 3.1.4. *Let $\mathcal{U}$ be an ω-cover of a space X. Then,*

$$\mathfrak{U}=\left\{\langle U\rangle_{\mathcal{F}(X)}:U\in\mathcal{U}\right\}$$

is an ω-cover of $\mathcal{F}(X)$.

Proof. Suppose that $\mathcal{U}$ is an ω-cover of X and $\mathcal{A}$ is finite set in $\mathcal{F}(X)$. Since each member of $\mathcal{A}$ is a finite set in X, $\bigcup\mathcal{A}$ is a finite set in X. Because $\mathcal{U}$ is an ω-cover of X, there exists $U_{\mathcal{A}}\in\mathcal{U}$ such that $\bigcup\mathcal{A}\subset U_{\mathcal{A}}$. This implies that

$$A\subset\bigcup\mathcal{A}\subset U_{\mathcal{A}}\text{ for every }A\in\mathcal{A}.$$

It shows that

$$A\in\langle U_{\mathcal{A}}\rangle_{\mathcal{F}(X)}\text{ for every }A\in\mathcal{A}.$$

Thus, $\mathcal{A}\subset\langle U_{\mathcal{A}}\rangle_{\mathcal{F}(X)}$. Hence, $\mathfrak{U}$ is an ω-cover of $\mathcal{F}(X)$.

Lemma 3.1.5 ([9])**.** *Let X be a space. If $\{x_1,\dots,x_r\}$ is a point of $\mathcal{F}(X)$ and*

$$\{x_1,\dots,x_r\}\in\langle U_1,\dots,U_s\rangle_{\mathcal{F}(X)},$$

then for each $j\le r$, we let

$$U_{x_j}=\bigcap\left\{U\in\{U_1,\dots,U_s\}:x_j\in U\right\}.$$

Observe that $\langle U_{x_1},\dots,U_{x_r}\rangle_{\mathcal{F}(X)}\subset\langle U_1,\dots,U_s\rangle_{\mathcal{F}(X)}$.

Lemma 3.1.6. *Let X be a space and $\mathfrak{U}$ be an ω-cover of $\mathcal{F}(X)$. If $F=\{x_1,\dots,x_r\}\in\mathcal{F}(X)$, then there exist $\mathcal{U}_F\in\mathfrak{U}$ and V_i is an open neighborhood of x_i in X for every $i\in\{1,\dots,r\}$ such that if $\{W_1,\dots,W_k\}$ is a subfamily of $\{V_1,\dots,V_r\}$, then*

$$\langle W_1,\dots,W_k\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_F.$$

Proof. Suppose that $\mathfrak{U}$ is an ω-cover and

$$F=\{x_1,\dots,x_r\}\in\mathcal{F}(X).$$

Then, $\langle F\rangle_{\mathcal{F}(X)}$ is a finite subset of $\mathcal{F}(X)$. Because $\mathfrak{U}$ is an ω-cover of $\mathcal{F}(X)$, there exists $\mathcal{U}_F\in\mathfrak{U}$ such that

$$\langle F\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_F.$$

This implies that if $\{y_1,\dots,y_s\}\in\langle F\rangle_{\mathcal{F}(X)}$, then there exist open subsets $U_1,\dots,U_j$ of X such that

$$\{y_1,\dots,y_s\}\in\left\langle U_1,\dots,U_j\right\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_F.$$

By Lemma 3.1.5, we can find open subsets $U_{y_1},\dots,U_{y_s}$ of X such that $y_j\in U_{y_j}$ for every $j\in\{1,\dots,s\}$ and

$$\{y_1,\dots,y_s\}\in\left\langle U_{y_1},\dots,U_{y_s}\right\rangle_{\mathcal{F}(X)}\subset\left\langle U_1,\dots,U_j\right\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_F.$$

Now, for each $i\in\{1,\dots,r\}$, we put

$$V_i=\bigcap\left\{U_{y_j}:\{y_1,\dots,y_s\}\in\langle F\rangle_{\mathcal{F}(X)},x_i\in U_{y_j},j\le s,s\in\mathbb{N}\right\}.$$

Then, V_i is an open neighborhood of x_i in X and if $\{W_1,\dots,W_k\}$ is a subfamily of $\{V_1,\dots,V_r\}$, then

$$\langle W_1,\dots,W_k\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_F.$$

Theorem 3.1.7. *Let X be a space. Then, X is an ω-Lindelöf space if and only if so is $\mathcal{F}(X)$.*

Proof. *Necessity*. Assume that $\mathfrak{U}$ is an ω-cover of $\mathcal{F}(X)$ and

$$F=\left\{x_F^{(1)},\dots,x_F^{(r_F)}\right\}\in\mathcal{F}(X).$$

By Lemma 3.1.6, there exist $\mathcal{U}_F\in\mathfrak{U}$ and $V_F^{(i)}$ is an open neighborhood of $x_F^{(i)}$ for every $i\in\{1,\dots,r_F\}$ in X such that if $\{W_1,\dots,W_k\}$ is a subfamily of $\left\{V_F^{(1)},\dots,V_F^{(r_F)}\right\}$, then

$$\langle W_1,\dots,W_k\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_F.$$

Then, $$\left\{\bigcup_{i=1}^{r_F}V_F^{(i)}:F=\left\{x_F^{(1)},\dots,x_F^{(r_F)}\right\}\in\mathcal{F}(X)\right\}$$

is an ω-cover of X. Since X is an ω-Lindelöf space, there exists $\{F_m:m\in\mathbb{N}\}$ is a sequence of members of $\mathcal{F}(X)$ such that

$$\left\{\bigcup_{i=1}^{r_{F_m}}V_{F_m}^{(i)}:m\in\mathbb{N}\right\}$$

is an ω-cover of X. Now, we will prove that $\left\{\mathcal{U}_{F_m}:m\in\mathbb{N}\right\}$ is an ω-cover of $\mathcal{F}(X)$. In fact, let $\mathcal{A}$ be a finite subset of $\mathcal{F}(X)$. Then, $\bigcup\mathcal{A}$ is a finite subset of X. This implies that there exists $m_{\mathcal{A}}\in\mathbb{N}$ such that

$$\bigcup\mathcal{A}\subset\bigcup_{i=1}^{r_{F_{m_{\mathcal{A}}}}}V_{F_{m_{\mathcal{A}}}}^{(i)}.$$

Therefore, for each $A\in\mathcal{A}$, there exists $\left\{W_A^{(1)},\dots,W_A^{k_A}\right\}$ is a subfamily of $\left\{V_{F_{m_{\mathcal{A}}}}^{(1)},\dots,V_{F_{m_{\mathcal{A}}}}^{(r_{F_{m_{\mathcal{A}}}})}\right\}$ such that

$$A\in\langle W_A^{(1)},\dots,W_A^{(k_A)}\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_{F_{m_{\mathcal{A}}}}.$$

It shows that $\mathcal{A}\in\mathcal{U}_{F_{m_{\mathcal{A}}}}$.

Thus, $\left\{\mathcal{U}_{F_m}:m\in\mathbb{N}\right\}$ is an ω-cover of $\mathcal{F}(X)$. Hence, $\mathcal{F}(X)$ is an ω-Lindelöf space.

Sufficiency. Suppose that $\mathcal{F}(X)$ is an ω-Lindelöf space. It follows from Lemma 3.1.2 and Remark 2.1.1(1) that $\mathcal{F}_1(X)$ is an ω-Lindelöf space. By Remark 2.1.1(2), the proof of sufficiency is completed.

Theorem 3.1.8. *Let X be a space. Then, X is an ω-Menger (resp., ω-Rothberger) space if and only if so is $\mathcal{F}(X)$.*

Proof. Necessity. Assume that $\left\{\mathfrak{U}^{(n)}:n\in\mathbb{N}\right\}$ is a sequence of ω-covers of $\mathcal{F}(X)$ and

$$F=\left\{x_F^{(1)},\dots,x_F^{(r_F)}\right\}\in\mathcal{F}(X).$$

By Lemma 3.1.6, for each $n\in\mathbb{N}$, there exist $\mathcal{U}_F^{(n)}\in\mathfrak{U}^{(n)}$ and $U_F^{(n,i)}$ is an open neighborhood of $x_F^{(i)}$ in X for every $i\in\{1,\dots,r_F\}$ such that if $\{V_1,\dots,V_k\}$ is a subfamily of $\{U_F^{(n,1)},\dots,U_F^{(n,r_F)}\}$, then

$$\langle V_1,\dots,V_k\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_F^{(n)}.$$

For each $n\in\mathbb{N}$, we put

$$\mathcal{W}_n=\left\{\bigcup_{i=1}^{r_F}U_F^{(n,i)}:F=\left\{x_F^{(1)},\dots,x_F^{(r_F)}\right\}\in\mathcal{F}(X)\right\}.$$

It is easy to see that $\{\mathcal{W}_n:n\in\mathbb{N}\}$ is a sequence of ω-covers of X.

Case 1. If X is an ω-Menger space, then for each $n\in\mathbb{N}$, there exists $\mathcal{Z}_n$ is a finite subset of $\mathcal{F}(X)$ such that

$$\bigcup_{n\in\mathbb{N}}\left\{\bigcup_{i=1}^{r_F}U_F^{(n,i)}:F\in\mathcal{Z}_n\right\}$$

is an ω-cover of X. Now, we will prove that

$$\bigcup_{n\in\mathbb{N}}\left\{\mathcal{U}_F^{(n)}:F\in\mathcal{Z}_n\right\}$$

is ω-cover of $\mathcal{F}(X)$. In fact, let $\mathcal{A}$ be a finite subset of $\mathcal{F}(X)$. Then, $\bigcup\mathcal{A}$ is a finite subset of X. This implies that there exist $n_{\mathcal{A}}\in\mathbb{N}$ and $F_{n_{\mathcal{A}}}\in\mathcal{Z}_{n_{\mathcal{A}}}$ such that

$$\bigcup\mathcal{A}\subset\bigcup_{i=1}^{r_{F_{n_{\mathcal{A}}}}}U_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}},i)}.$$

It shows that for each $A\in\mathcal{A}$, there exists a subfamily $\left\{V_A^{(1)},\dots,V_A^{(k_A)}\right\}$ of $\left\{U_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}},1)},\dots,U_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}},r_{F_{n_{\mathcal{A}}}})}\right\}$ such that

$$A\in\left\langle V_A^{(1)},\dots,V_A^{(k_A)}\right\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}})}.$$

Thus, $\mathcal{A}\subset\mathcal{U}_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}})}$.

Therefore, $\bigcup_{n\in\mathbb{N}}\left\{\mathcal{U}_F^{(n)}:F\in\mathcal{Z}_n\right\}$

is an ω-cover of $\mathcal{F}(X)$. Hence, $\mathcal{F}(X)$ is an ω-Menger space.

Case 2. If X is an ω-Rothberger space, then for each $n\in\mathbb{N}$, there exists $F_n\in\mathcal{F}(X)$ such that

$$\left\{\bigcup_{i=1}^{r_{F_n}}U_{F_n}^{(n,i)}:n\in\mathbb{N}\right\}$$

is an ω-cover of X. Now, we will prove that $\left\{\mathcal{U}_{F_n}^{(n)}:n\in\mathbb{N}\right\}$ is an ω-cover of $\mathcal{F}(X)$. Indeed, let $\mathcal{A}$ be a finite subsets of $\mathcal{F}(X)$. Then, $\bigcup\mathcal{A}$ is an finite subset of X. This implies that there exists $n_{\mathcal{A}}\in\mathbb{N}$ such that

$$\bigcup\mathcal{A}\subset\bigcup_{i=1}^{r_{F_{n_{\mathcal{A}}}}}U_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}},i)}.$$

It shows that for each $A\in\mathcal{A}$, there exists a subfamily $\left\{V_A^{(1)},\dots,V_A^{(k_A)}\right\}$ of $\left\{U_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}},1)},\dots,U_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}},r_F)}\right\}$ such that

$$A\in\left\langle V_A^{(1)},\dots,V_A^{(k_A)}\right\rangle_{\mathcal{F}(X)}\subset\mathcal{U}_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}})}.$$

Thus, $\mathcal{A}\subset\mathcal{U}_{F_{n_{\mathcal{A}}}}^{(n_{\mathcal{A}})}$.

Therefore, $\left\{\mathcal{U}_{F_n}^{(n)}:n\in\mathbb{N}\right\}$

is an là ω-cover of $\mathcal{F}(X)$. Hence, $\mathcal{F}(X)$ is an ω-Rothberger space.

Sufficiency. Let $\mathcal{F}(X)$ be an ω-Menger (resp., ω-Rothberger) space. It follows from Lemma 3.1.3 and Remark 2.1.1(1) that $\mathcal{F}_1(X)$ is an ω-Menger (resp., ω-Rothberger) space. Thus, X is an ω-Menger (resp., ω-Rothberger) space by Remark 2.1.1(2).

Example 3.1.9. *An example of the equivalence of property ω-Menger (resp., ω-Rothberger, ω-Lindelöf) between space X and the hyperspace $\mathcal{F}(X)$.*

Solution. Let X be a countable set with the discrete topology.

Claim 1. X is an ω-Menger (resp. ω-Rothberger) space.

Assume that $\{\mathcal{U}_m:m\in\mathbb{N}\}$ is a sequence of ω-covers of X. Because X is countable,

$$\{F\subset X:F\text{ is finite}\}=\{F_k:k\in\mathbb{N}\}.$$

For each $k \in \mathbb{N}$, since $\mathcal{U}_k$ is an ω-cover of X, there exists $U_k \in \mathcal{U}_k$ such that

$$F_k \subset U_k.$$

Now, for each $m \in \mathbb{N}$, if we put $\mathcal{V}_m = \{U_m\}$, then $\mathcal{V}_m$ is a finite subfamily of $\mathcal{U}_m$ and $\bigcup_{m\in\mathbb{N}} \mathcal{V}_m$ is an ω-cover of X. Hence, X is an ω-Menger space.

Since $\{U_k : k \in \mathbb{N}\}$ is an ω-cover of X, X is an ω-Rothberger space.

Claim 2. X is an ω- Lindelöf space.

Assume that $\mathcal{U}$ is an ω-cover of X. Because X is countable,

$$\{F \subset X : F \text{ is finite}\} = \{F_k : k \in \mathbb{N}\}.$$

Since $\mathcal{U}$ is a ω-cover of X, for each $k \in \mathbb{N}$, there exists $U_k \in \mathcal{U}$ such that $F_k \subset U_k$. Therefore, $\{U_k : k \in \mathbb{N}\}$ is an ω-cover of X. Hence, X is ω- Lindelöf space.

Claim 3. $\mathcal{F}(X)$ is an ω-Menger (resp. ω-Rothberger) space.

Suppose that $\{\mathfrak{U}_m : m \in \mathbb{N}\}$ is a sequence of ω-covers of $\mathcal{F}(X)$. Because X is countable, $\mathcal{F}(X)$ is countable. This implies that

$$\{\mathcal{A} \subset \mathcal{F}(X) : \mathcal{A} \text{ is finite}\} = \{\mathcal{A}_k : k \in \mathbb{N}\}.$$

For each $k \in \mathbb{N}$, since $\mathfrak{U}_k$ is an ω-cover of $\mathcal{F}(X)$, there exists $\mathcal{V}_k \in \mathfrak{U}_k$ such that

$$\mathcal{A}_k \subset \mathcal{V}_k.$$

Now, for each $m \in \mathbb{N}$, if we put $\mathfrak{V}_m = \{\mathcal{V}_m\}$, then $\mathfrak{V}_m$ is a finite subfamily of $\mathfrak{U}_m$ and $\bigcup_{m\in\mathbb{N}} \mathfrak{V}_m$ is an ω-cover of X. Hence, $\mathcal{F}(X)$ is an ω-Menger space.

Since $\{\mathcal{V}_k : k \in \mathbb{N}\}$ is an ω-cover of $\mathcal{F}(X)$, $\mathcal{F}(X)$ is an ω-Rothberger space.

Claim 4. $\mathcal{F}(X)$ is an ω-Lindelöf space.

Suppose that $\mathfrak{U}$ is an of ω-cover of $\mathcal{F}(X)$. Because X is countable, $\mathcal{F}(X)$ is countable. This implies that

$$\{\mathcal{A} \subset \mathcal{F}(X) : \mathcal{A} \text{ is finite}\} = \{\mathcal{A}_k : k \in \mathbb{N}\}.$$

Since $\mathfrak{U}$ is an ω-cover of $\mathcal{F}(X)$, there exists $\mathcal{V}_k \in \mathfrak{U}$ such that $\mathcal{A}_k \subset \mathcal{V}_k$. Therefore, $\{\mathcal{V}_k : k \in \mathbb{N}\}$ is an ω-cover of $\mathcal{F}(X)$. Hence, $\mathcal{F}(X)$ is an ω-Lindelöf space.

Comparison: In [13], J. C. Rosa et al. demonstrated that a space X has the strongly star-Menger property if and only if the space $\mathcal{F}(X)$ has the strongly star-Rothberger. In this paper, we extend their results by proving that X is an ω-Menger (resp., ω-Rothberger) space if and only if $\mathcal{F}(X)$ is as well. Additionally, we show that X is ω-Lindelöf if and only if $\mathcal{F}(X)$ satisfies the same condition.

3.2. Discussion

The main results of this paper are presented in Lemmas 3.1.1, 3.1.2, 3.1.3, 3.1.4, 3.1.5 and Theorems 3.1.7, 3.1.8.

- Lemmas 3.1.1, 3.1.2, 3.1.3, 3.1.4, 3.1.5 and 3.1.6 are used to prove Theorems 3.1.7 and 3.1.8.

- Theorems 3.1.7 and 3.1.8 demonstrate the equivalence of the ω- Lindelöf (resp., ω-Menger, ω-Rothberger) property between the topological space X and the Vietoris hyperspace $\mathcal{F}(X)$.

4. Conclusion

In this paper, the authors present and provide detailed proofs of some new results concerning the equivalence of ω-cover and certain properties defined by ω-cover between the topological space X and the Vietoris hyperspace $\mathcal{F}(X)$. These results contribute to enriching the field of research on generalized metric properties in general topology.

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