STOCHASTIC VIBRATION ANALYSIS OF BEAM BRIDGES SUBJECTED TO MOVING LOADS WITH UNCERTAIN PARAMETERS USING THE PROBABILITY DENSITY EVOLUTION METHOD

PHÂN TÍCH DAO ĐỘNG CỦA CẦU DẦM CHỊU TẢI TRỌNG DI ĐỘNG VỚI CÁC THAM SỐ ĐẦU VÀO NGẪU NHIÊN BẰNG PHƯƠNG PHÁP TIẾN HÓA MẬT ĐỘ XÁC SUẤT

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Abstract - This paper proposes an effective approach for stochastic vibration analysis of beam bridges subjected to moving vehicular loads, with consideration of uncertainties in both axle load and vehicle speed. The study is conducted on the Suoi Tuong bridge using a finite element model. The loads are collected from 223 actual vehicle samples, and vehicle speed is modeled as an independent normally distributed random variables. The Probability Density Evolution Method (PDEM) is employed to simulate the time-dependent evolution of the probability density function of the dynamic response. Two approaches are used to construct the input probability space: one based on kernel density estimation (KDE) and the other on discrete empirical samples. The results show that, PDEM provides high accuracy and flexibility in handling experimental data, making it suitable for reliability assessment, exceedance analysis, and structural health monitoring of bridges under uncertainty.

Key words - Stochastic vibration; beam bridge; Probability Density Evolution Method; weigh-in-motion load; random vehicle velocity

Tóm tắt - Bài báo đề xuất một phương pháp hiệu quả trong phân tích dao động ngẫu nhiên của cầu dầm chịu tải trọng di động, có xét đến tính bất định của tải trọng và vận tốc xe. Nghiên cứu được thực hiện trên cầu Suối Tượng, với mô hình phần tử hữu hạn. Tải trọng được thu thập từ 223 mẫu xe thực tế, vận tốc được mô hình hóa là biến ngẫu nhiên phân bố chuẩn, độc lập với tải trọng. Phương pháp tiến hóa mật độ xác suất (PDEM) được áp dụng để mô phỏng sự thay đổi theo thời gian của phân bố xác suất phản ứng dao động. Hai cách tiếp cận để xây dựng không gian xác suất đầu vào, bao gồm ước lượng mật độ xác suất liên tục từ dữ liệu (KDE) và sử dụng tập mẫu thực nghiệm rời rạc. Kết quả cho thấy, PDEM đạt độ chính xác cao và khả năng linh hoạt trong xử lý dữ liệu thực nghiệm, phù hợp trong các bài toán đánh giá độ tin cậy, phân tích vượt ngưỡng và quan trắc kết cấu cầu trong điều kiện bất định.

Từ khóa - Dao động ngẫu nhiên; cầu dầm; phương pháp tiến hóa mật độ xác suất; tải trọng trạm cân; vận tốc ngẫu nhiên

1. Introduction

In the dynamic analysis of bridge structures, especially beam bridges subjected to moving loads, uncertainties such as vehicle loads, traffic speed, wheel–deck contact conditions, and material parameters directly affect the structural vibration response. These factors cannot be fully described by deterministic values and must be considered as random variables or random fields, particularly in the context of increasingly complex traffic and the growing availability of operational data.

Three main approaches have been developed to address the problem of stochastic vibration analysis: the Monte Carlo Simulation (MCS) method, the Stochastic Finite Element Method (SFEM), and the Probability Density Evolution Method (PDEM). MCS is a widely used method that employs random sampling and statistical analysis to estimate the probabilistic characteristics of structural responses. However, this method requires a large number of samples, which becomes particularly expensive for nonlinear systems or time-domain analyses [1]. SFEM partially overcomes this drawback by expanding random fields using basis functions (such as Karhunen–Loève or Polynomial Chaos), thereby propagating input uncertainties to the system response [2], [3]. Nevertheless, SFEM encounters difficulties when dealing with non-Gaussian probability density functions or time-dependent dynamic problems [4].

PDEM, developed by J. Li and J. Chen [5], is considered a significant advancement in the stochastic vibration analysis of structures. This method establishes the Probability Density Evolution Equation (PDEE) based on the principle of probability conservation, enabling direct description of the time evolution of the probability density function of the structural response without the need for sampling (as in MCS) or expansion (as in SFEM). PDEM is particularly suitable for high-degree-of-freedom, nonlinear vibration problems with complex random input variables [6].

Recently, PDEM has been extended and effectively applied to various structural problems. Mao et al. [7, 8] implemented PDEM in high-speed railway bridge–track systems, incorporating uncertainties such as elastic modulus, vehicle loads, and velocities. More recent studies have integrated PDEM with machine learning and Bayesian optimization for the stochastic vibration analysis of magnetic levitation railway systems [8]. In Vietnam, Dang Cong Thuat et al. [9, 10] have applied PDEM to the stochastic vibration analysis of structures subjected to earthquakes, providing a basis for structural reliability assessment. These studies demonstrate the scalability and applicability of PDEM to the analysis of structural vibrations under complex dynamic loading.

Based on this foundation, the present study proposes the application of the PDEM to analyze the stochastic vibration of the Suoi Tuong reinforced concrete beam bridge, using weigh-in-motion (WIM) data, including vehicle loads and speeds. Two approaches to modeling the probability space are considered: (1) employing a kernel density estimation (KDE) to construct the empirical probability density function; and (2) treating discrete samples as uniformly distributed random variables. The results obtained from PDEM will be validated against MCSs to assess their accuracy, convergence behavior, and computational cost. The overall objective is to develop an efficient analytical framework capable of being applied in situations with limited experimental data.

2. Problem modeling and experimental data

2.1. Structural model of Suoi Tuong bridge

The reinforced concrete beam structure of Suoi Tuong Bridge, part of the Phan Thiet–Dau Giay highway project, is a single-span I-girder bridge subjected to the load of a three-axle vehicle. The bridge includes lateral connection systems and is modeled as shown in Figure 1.

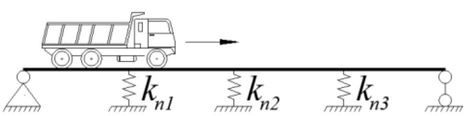

Figure 1. *Diagram of moving vehicle loads on the bridge*

The cross-section of the bridge comprises 11 reinforced concrete I-beams, as shown in Figure 2. The main girders are connected laterally by a 0.2 m thick deck slab and five cross beams positioned at the two supports, 1/4L, 1/2L, and 3/4L.

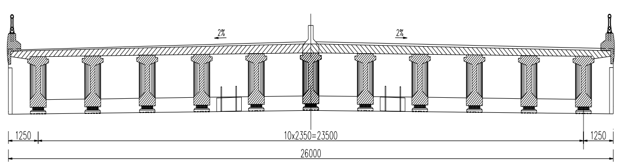

Figure 2. *Cross section of Suoi Tuong bridge*

The structural parameters of the bridge are determined as follows: E=3647550.59 T/m^2; J_d=0.5213m^4; F_d=1.085 m^2; ρF_d=2.8 T/m, internal friction coefficient θ = 0.027 and external friction coefficient β=0.01.

The three-axle vehicle used in the numerical simulation is a Foton truck with the following parameters: m21=0.26 T; m22=m23=0.87 T; k11=240 T/m; k12=k13=520 T/m; k21=240 T/m, k22=k23=380 T/m; d11=0.7344 Ts/m; d12=d13=0.3672 Ts/m; d21=0.4 Ts/m; d22=d23= 0.8 Ts/m.

The vehicle–bridge interaction is modeled using the KC-05 finite element software, and the steps for dynamic response analysis are detailed in references [11,12].

In the dynamic simulations, the quantities of interest in this study are the vibrations, including displacement and velocity at midspan, which is typically the location of maximum amplitude due to the dynamic effect of moving loads.

2.2. Load and speed data

The load data were collected from the weigh-in-motion system managed by Road Management Zone IV, comprising 223 vehicle load samples, represented as the total load (including vehicle and cargo) [13, 14]. Each sample corresponds to a unique vehicle, with the time, weight, and vehicle type recorded, ensuring representativeness of actual traffic flow.

Due to the lack of direct speed measurements, vehicle speed is assumed to be an independent, normally distributed random variable, with statistical parameters (mean and standard deviation) calibrated based on actual operating conditions at the bridge site. The load P and speed v are considered as two independent random variables, accurately reflecting the experimental nature and facilitating the modeling of the two-dimensional probability space.

2.3. Stochastic modeling of input data

The two random input variables are processed using two approaches for probability space construction:

- Approach 1: The probability density function of P is estimated using the Kernel Density Estimation (KDE) method, allowing continuous probabilistic characterization from discrete data. Combined with the normal distribution of v, the (P, v) probability space is discretized into a 20 × 20 grid.
- Approach 2: Each vehicle in the dataset is considered as a discrete random variable with uniform probability, denoted as Xe_i. The probability space is thus constructed from the combination of 223 vehicle types with 20 speed values, resulting in a discrete grid of 223 × 20 points. This approach reflects the practical scenario where the bridge is subjected to a sequence of heterogeneous vehicles and is flexible for problems with limited data.

Both approaches are incorporated into the analytical model to verify the accuracy and stability of the PDEM under different modeling conditions for the input variables.

3. Probability Density Evolution Method (PDEM)

The PDEM is an advanced computational approach for stochastic vibration analysis of structures, particularly suitable when input parameters such as load and velocity exhibit uncertainty. Unlike sampling-based methods such as MCS, or statistical approximation methods like the SFEM, PDEM tackles the problem by formulating a system of differential equations that directly describe the time evolution of the probability density function (PDF) of the structural response.

By integrating the dynamic model of the structural system with the PDEE, this method enables the determination of the response probability distribution in the state domain without requiring numerous individual simulations. This is a significant advantage in practical problems, where computational cost is a limiting factor and experimental data are often incomplete or not repeatable.

This section presents the detailed structure and principles of the PDEM [5,6], including:

- Formulation of the state equation describing the dynamic response of the structural system under the influence of random variables;

- Construction of the PDEE for the structural response;
- Presentation of the numerical algorithm and discretization techniques for the probability space.

These components provide the theoretical and computational foundation for applying PDEM to practical problems in the following sections, especially the stochastic vibration analysis of Suoi Tuong Bridge using field measurement data.

3.1. State Differential equation of vibration

The dynamic response of a bridge structure under random moving loads can be described by the following second-order differential equation system:

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t;\boldsymbol{Z}) + \boldsymbol{C}\dot{\boldsymbol{u}}(t;\boldsymbol{Z}) + \boldsymbol{K}\boldsymbol{u}(t;\boldsymbol{Z}) = \mathbf{f}(t;\boldsymbol{Z}) \quad (1)$$

where:

- $\boldsymbol{u}(t;\boldsymbol{Z}) \in R^n$: displacement vector at time t,
- $\boldsymbol{M},\boldsymbol{C},\boldsymbol{K} \in R^{n\times n}$: deterministic mass, damping, and stiffness matrices, respectively,
- $\mathbf{f}(t;\boldsymbol{Z}) \in R^n$: load vector dependent on the random variable $\boldsymbol{Z} \in R^{n_z}$,
- n: number of degrees of freedom of the system.

To facilitate mathematical treatment and the formulation of the PDEE, equation (1) is transformed into a first-order differential equation system by introducing a state vector as follows:

$$X(t;\boldsymbol{Z}) = \begin{bmatrix}\boldsymbol{u}(t;\boldsymbol{Z})\\ \dot{\boldsymbol{u}}(t;\boldsymbol{Z})\end{bmatrix} \in R^{2n}$$

Thus, the first-order state equation can be written as:

$$\dot{\boldsymbol{X}}(t;\boldsymbol{Z}) = \boldsymbol{A}\,\boldsymbol{X}(t;\boldsymbol{Z}) + F(t;\boldsymbol{Z}) \quad (2)$$

with:

$$A = \begin{bmatrix}0 & \boldsymbol{I}\\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C}\end{bmatrix}, \boldsymbol{F}(t;\boldsymbol{Z}) = \begin{bmatrix}0\\ \boldsymbol{M}^{-1}\mathbf{f}(t;Z)\end{bmatrix}$$

This is the fundamental equation used to describe the dynamic state of the system and serves as the foundation for formulating the PDEE in the following section.

In this study, the response quantity of interest is a scalar function $U(t;\boldsymbol{Z}) \in R$, such as the vertical displacement at midspan. This function is extracted from the solution vector $\boldsymbol{u}(t;\boldsymbol{Z})$ via a linear projection or appropriate interpolation.

3.2. Probability Density Evolution Equation (PDEE)

To describe the time evolution of the probability distribution of the structural response $U(t;Z)$, the PDEE must be established. The formulation of this equation is based on the principle of probability conservation in the state space of the random system.

Consider the first-order differential equation presented in (2). Let $\boldsymbol{Y}(t) = \begin{bmatrix}\boldsymbol{X}(t;\boldsymbol{Z})\\ \boldsymbol{Z}\end{bmatrix} \in R^{2n+n_z}$ be the augmented state vector, combining both the dynamic response and the random input variables. Since $\boldsymbol{Z}$ is constant in time (does not vary within each sample), we have $\dot{\boldsymbol{Z}} = 0$, so the augmented dynamic equation is:

$$\dot{\boldsymbol{Y}}(t) = \boldsymbol{G}(\boldsymbol{Y},t) = \begin{bmatrix}\boldsymbol{A}\,\boldsymbol{X}(t;\boldsymbol{Z}) + \mathrm{F}(t;\boldsymbol{Z})\\ 0\end{bmatrix}$$

Denoting $p_{\boldsymbol{Y}}(\boldsymbol{y},t)$ as the probability density function (PDF) of the vector $\boldsymbol{Y}(t)$, the conservation of probability theorem is applied to formulate the evolution equation as follows:

$$\frac{\partial p_{\boldsymbol{Y}}(y,t)}{\partial t} + \nabla_y \cdot [p_{\boldsymbol{Y}}(\boldsymbol{y},t)\,\boldsymbol{G}(\boldsymbol{y},t)] = 0 \quad (3)$$

This is a first-order linear partial differential equation, a type of Liouville equation, describing the time evolution of the probability density of the entire random system.

Equation (3) usually has a very high order (dimension $2n + n_z$), making direct analytical solutions impractical. Therefore, for practical analysis, this equation is reduced to describe only the evolution of the scalar response $U(t;\boldsymbol{Z}) \in R$, by integrating out the irrelevant variables.

Assuming $p_{U,\boldsymbol{Z}}(\boldsymbol{u},\boldsymbol{z},t)$ is the joint probability density of $U(t;\boldsymbol{Z})$ and $\boldsymbol{Z}$, the PDF of the response is:

$$p_U(\boldsymbol{u},t) = \int_{R^{n_z}} p_{U,\boldsymbol{Z}}(\boldsymbol{u},\boldsymbol{z},t)\,d\boldsymbol{z} \quad (4)$$

Formulating the evolution equation for $p_{U,Z}$ can be achieved by projecting equation (3) and then solving it numerically, as will be described in Section 3.3.3.

The initial condition for equation (3) is determined by the initial deterministic value of the system (i.e., at $t = 0$) and the probability distribution of Z:

$$p_{\boldsymbol{Y}}(\boldsymbol{y},0) = \delta(\boldsymbol{X} - \boldsymbol{X}_0)\cdot p_{\boldsymbol{Z}}(\boldsymbol{z}) \quad (5)$$

where, $\delta(\cdot)$ is the Dirac delta function and $\boldsymbol{X}_0$ is the deterministic initial condition of the system at the initial time.

The PDEE (3), together with the initial condition (5), forms the basis for determining the evolution of the probability of the structural response over time. This is the core foundation of the PDEM in this study.

3.3. Numerical algorithm and discretization of the probability space

The PDEE presented in (3) is a first-order linear partial differential equation of the form:

$$\frac{\partial p_{\boldsymbol{Y}}(\boldsymbol{y},t)}{\partial t} + \nabla_{\boldsymbol{y}} \cdot [p_{\boldsymbol{Y}}(\boldsymbol{y},t)\,\boldsymbol{G}(\boldsymbol{y},t)] = 0$$

Given the high dimensionality of the state space $\boldsymbol{y} \in R^{2n+n_z}$, direct analytical solutions are infeasible for most practical engineering problems. Instead, numerical methods, specifically the Finite Difference Method (FDM) combined with the Total Variation Diminishing (TVD) scheme, are used to ensure the accuracy and stability of the solution.

3.3.1. Discretization of the probability space

The probability space of the input random variables $Z \in R^{n_z}$ is discretized into a set of grid points $\{z_q\}_{q=1}^N$. Two strategies are employed for discretization:

- For continuous variables with a well-defined probability density function (such as v or P after smoothing via KDE): the value domain of each variable is divided into discrete nodes $z_{j,i}$ with associated weights $\bar{p}_Z(z_{j,i})$, where $j = 1, \dots, n_z$ and $i = 1, \dots, N_j$. The discrete domain is constructed as a tensor grid:

$$\boldsymbol{z}_q = \left(z_{1,i_1}, z_{2,i_2}, \dots, z_{n_z,i_{n_z}}\right), q = 1, \dots, N = \prod_{j=1}^{n_z} N_j$$

• For empirical discrete random variables (e.g., the set of vehicle samples collected from the weigh station): each occurrence of vehicle Xe_i with corresponding weight P_i is treated as an independent sample with equal probability, i.e., uniformly distributed over the set $\{P_1, P_2, \dots, P_{223}\}$.

The random variable v in both approaches is assumed to be a continuous, normally distributed variable, independent of P, allowing for separate discretization grids.

3.3.2. Numerical solution of the PDEE

The PDEE is solved using the Total Variation Diminishing (TVD) finite difference scheme, a specialized technique within the finite difference method to prevent spurious oscillations in the solution of first-order partial differential equations. Specifically:

- Equation (3) is decomposed along each dimension in the state space.

- The convective term is handled using the Lax–Wendroff scheme combined with the Roe–Sweby flux limiter to ensure stability and non-negativity of the density function.

- The Courant–Friedrichs–Lewy (CFL) condition is checked to ensure the stability of the scheme at each time step Δt.

3.3.3. Estimation of the response density function

After solving the evolution equation for the joint probability density $p_{U,Z}(u, z, t)$, the response density is determined by marginal integration:

$$p_U(u,t) = \int_{R^{n_z}} p_{U,\mathbf{Z}}(u, \mathbf{z}, t)\, d\mathbf{z} \approx \sum_{q=1}^{N} \Delta z_q\, p_{U,\mathbf{Z}}(u, \mathbf{z}_q, t)$$

where, Δz_q is the integration weight (volume element) for the q grid point in the discretized probability space. The value $p_{U,\mathbf{Z}}(u, \mathbf{z}_q, t)$ is obtained by solving the PDEE for the input condition $\mathbf{z}_q$, and then linearly interpolated with respect to the response variable u.

4. Analysis results

4.1. Comparison with the Monte Carlo Method

The Monte Carlo method is used as a reference, with 10,000 random samples generated from the empirical distribution of the load (estimated using the KDE method) and the vehicle speed (modeled as a normal distribution). For each sample, the dynamic analysis is solved independently in the time domain to collect structural response data.

The probability distribution of the response (vertical displacement at midspan) is constructed from the Monte Carlo sample set using a histogram and compared with the results obtained from the PDEM at selected time instants. Two main statistical quantities are considered:

• The expectation $E[U(t)]$;

• The standard deviation $Std[U(t)]$.

The analysis results show that the PDEM achieves very high accuracy compared to the Monte Carlo method, with the absolute error of the mean and standard deviation of the midspan displacement being less than 2%. In terms of computational cost, PDEM demonstrates a significant advantage by substantially reducing the number of deterministic dynamic analyses required to achieve equivalent convergence.

Specifically, in Approach 1 - using a continuous distribution estimated from experimental data via KDE - PDEM only needs to solve about 400 dynamic problems, corresponding to the combination of 20 load values and 20 velocity values. In Approach 2 - directly applying 223 discrete load samples combined with 20 velocity values - the total number of analyses is 4,460. Meanwhile, to achieve similar accuracy, the Monte Carlo method requires at least 10,000 simulations.

These results indicate that PDEM not only provides high accuracy in stochastic response prediction, but also offers a clear advantage in computational efficiency. The ability to simultaneously process both continuous and discrete experimental data also demonstrates the method's flexibility for practical applications, especially under conditions of limited data or non-standard distributions.

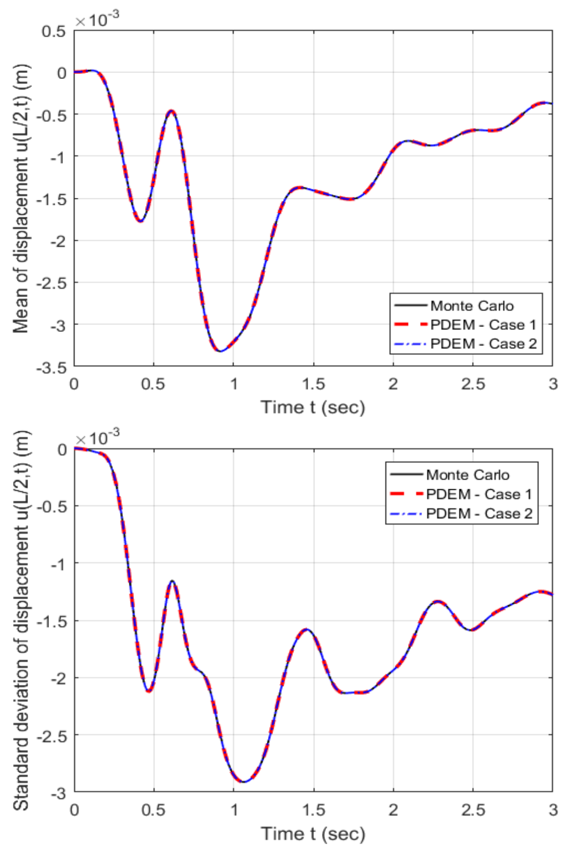

Figure 3. *Comparison of results between Monte Carlo and PDEM*

4.2. Evolution of the probability density of structural response over time

A notable advantage of the PDEM is its ability to simulate the entire evolution process of the probability density function $p_U(u, t)$ over time, rather than providing statistics only at the final time, as in Monte Carlo.

Figures 4 and 5 illustrate the time evolution of the probability density function of the vibration response p_U (u,t) at the midspan of Suoi Tuong Bridge under the effect of random moving loads. At early time instances (Figure 4, with t=0.1, 0.15 and 0.2 seconds), the probability density exhibits a single peak, symmetric shape, closely resembling a normal distribution - reflecting a relatively stable vibration state and a clear probabilistic law. However, as time increases (Figure 5, with t=0.5, 1.0 and 1.5 seconds), the density function evolves into a more complex form, with multiple peaks, asymmetry, and a distribution that no longer follows a standard normal law. This indicates that the cumulative effects of random factors such as load and vehicle speed have altered the probabilistic nature of the structural response, causing the vibration process to become nonlinear and irregular over time.

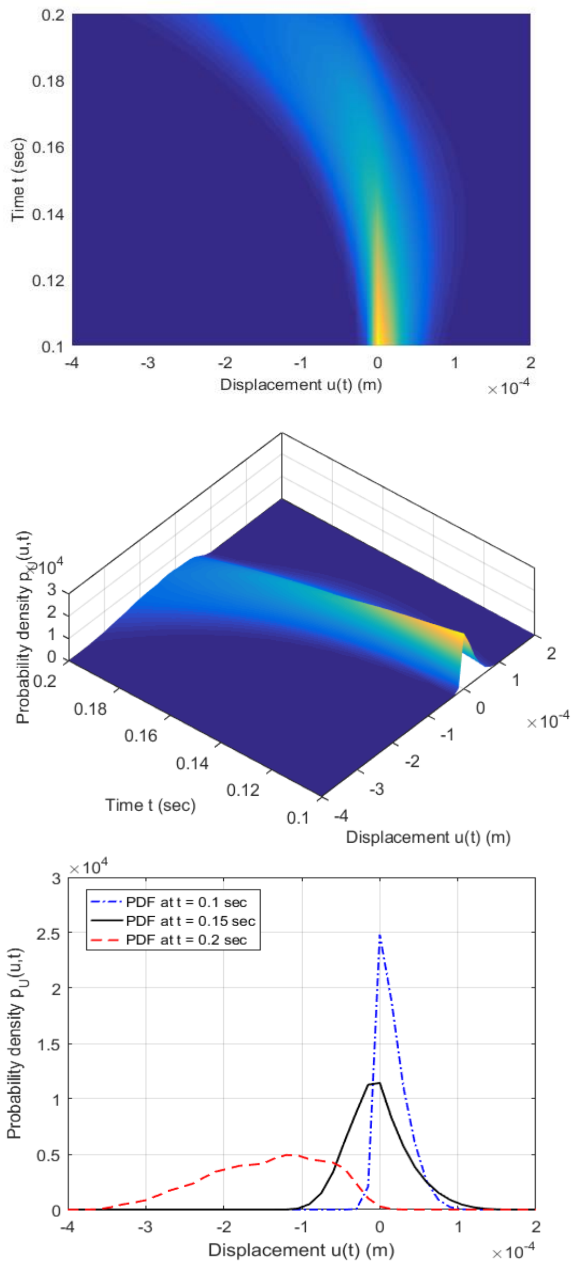

Figure 4. *The probability density $p_U(u,t)$ is determined at three characteristic times $t = 0.1s$, $0.15s$ and $0.2s$*

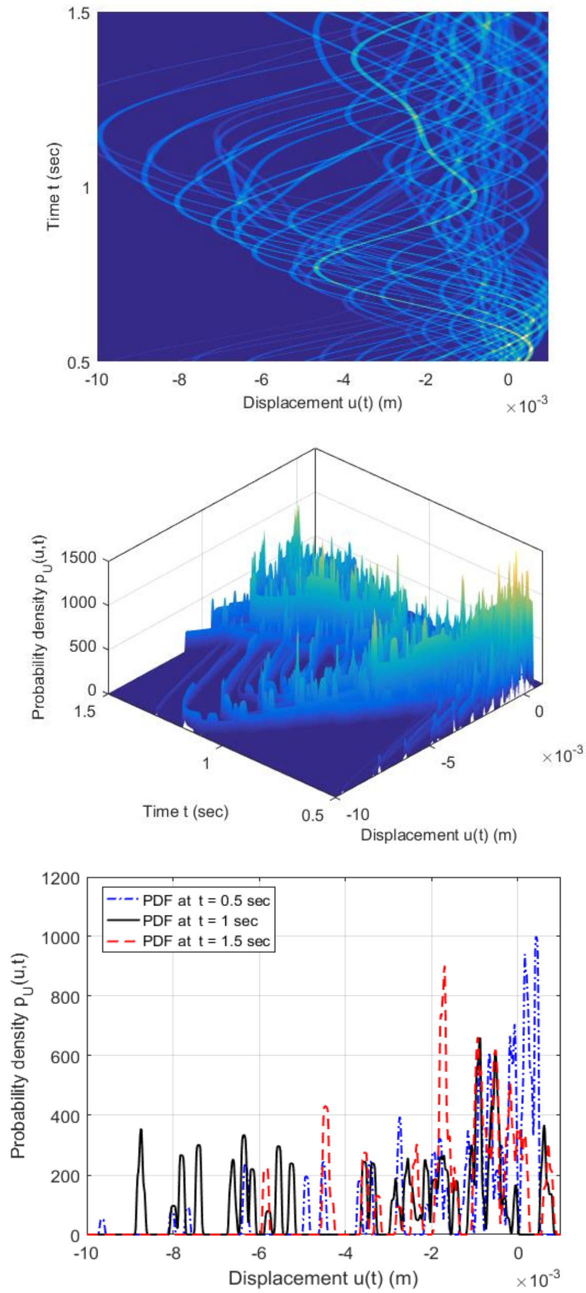

Figure 5. *The probability density $p_U(u,t)$ is determined at three times $t = 0.5s$, $1.0s$ and $1.5s$*

5. Conclusion

This paper has presented a systematic study of the stochastic vibration analysis of beam bridges subjected to moving loads under uncertainty, using experimental data from weigh-in-motion stations and applying the PDEM. Unlike traditional statistical methods such as Monte Carlo or the SFEM, PDEM allows for direct modeling of the evolution process of the probability density function of the structural response over time, thereby quantifying the probability distribution continuously and comprehensively in the space–time domain.

The study utilized 223 actual vehicle load samples and modeled vehicle speed as a normally distributed random variable, independent of the load. Two approaches to constructing the input probability space were implemented: (1) estimating the continuous probability density function using the KDE technique; and (2) directly using discrete data with the assumption of uniform distribution. Both were integrated into the PDEM framework for response analysis.

The results show that PDEM accurately predicts the mean and standard deviation of midspan displacement, with an error of less than 2–3% compared to Monte Carlo results, while requiring only 400 to 4,460 dynamic simulations (compared to 10,000 for Monte Carlo). This confirms the computational efficiency advantage of PDEM, especially for large-scale or strongly nonlinear engineering problems.

Beyond providing first- and second-order statistical quantities, PDEM also enables direct simulation of the evolution process of the response probability, an essential feature for problems involving threshold exceedance analysis, real-time reliability assessment, and probabilistic optimization. The simultaneous verification of both probability input modeling approaches also demonstrates the flexibility and high applicability of PDEM, even when measurement data are limited.

From these results, it can be concluded that PDEM is an efficient, accurate, and suitable method for stochastic vibration analysis under real-world conditions, where loads are not only uncertain but also originate from field data. This study also opens up prospects for applying PDEM to other infrastructure systems such as cable-stayed bridges, multi-span continuous bridges, or structures subjected to random environmental effects, serving probabilistic design, safety assessment, and risk management objectives.

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