SHEAR DEFORMATION THEORIES WITH F(Z) FUNCTION AND ITS APPLICATIONS

LÝ THUYẾT BIẾN DẠNG CẮT VỚI HÀM F(Z) VÀ CÁC ỨNG DỤNG CỦA NÓ

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(Received: June 06, 2025; Revised: April 03, 2026; Accepted: May 08, 2026)

DOI: 10.31130/ud-jst.2026.24(5A).302E

Abstract - Shear deformation theories (SDTs) play a critical role in the accurate modeling of thick beams and plates, where transverse shear deformation has a non-negligible influence on the overall mechanical response. Traditional structural theories, such as the classical plate theory (CPT) and first-order shear deformation theory (FSDT), often fail to capture the true nature of shear strain distribution, especially in moderately thick and thick structures, due to oversimplified assumptions. To overcome these limitations, higher-order shear deformation theories (HSDTs) have been proposed, which incorporate more refined representations of shear strain through the introduction of a shear strain distribution function, typically denoted as f(z). This article thoroughly examines shear deformation theories that include an f(z) function, including their mathematical formulations, advantages, applications, and potential research directions.

Key words - Shear deformation; thick plates; HSDT; strain function; structural modeling

Tóm tắt - Các lý thuyết biến dạng cắt (SDT) rất quan trọng đối với việc mô hình hóa các dầm và tấm dày vì biến dạng cắt ngang ảnh hưởng đáng kể đến ứng xử cơ học của kết cấu. Các mô hình thông thường, như lý thuyết tấm cổ điển (CPT) và lý thuyết biến dạng cắt bậc nhất (FSDT), thể hiện những hạn chế trong việc biểu diễn chính xác các hiệu ứng cắt, đặc biệt là trong các kết cấu có độ dày hoặc dày vừa phải, do các giả định quá đơn giản. Để giải quyết các hạn chế này, các lý thuyết biến dạng cắt bậc cao (HSDT) đã được xây dựng để giải quyết những thách thức này, sử dụng hàm phân phối biến dạng cắt f(z) để tăng cường dự đoán ứng suất cắt. Bài viết này đánh giá các lý thuyết biến dạng cắt tích hợp hàm f(z), bao gồm các công thức toán học, lợi ích, ứng dụng và các hướng nghiên cứu triển vọng của chúng.

Từ khóa - Biến dạng cắt; tấm dày; biến dạng cắt bậc cao; hàm biến dạng; mô hình hóa kết cấu

1. Introduction

Classical structural theories argue that plane sections retain their planar shape and stay perpendicular to the mid-plane post-deformation [1], [2]. This assumption is effective for thin plates and beams, but results in inaccurate stress and deformation predictions for thick structures owing to transverse shear effects.

The first-order shear deformation theory (FSDT) assumes a uniform transverse shear strain distribution, which fails to meet the zero shear stress boundary requirements at the upper and lower surfaces [3]. This scenario necessitates the introduction of an empirical shear correction factor to modify shear stiffness estimates, often leading to mistakes [4].

To address these restrictions, higher-order shear deformation theories (HSDTs) have been formulated, whereby shear strain is dependent on the thickness coordinate z [5], [6]. These theories seek to [7]:

- Enhance the precision of stress and displacement forecasts in thick beams and plates.

- Eliminate the need for shear correction factors by inherently fulfilling the shear stress-free boundary criteria.

- Enhance modeling capabilities for advanced materials, including functionally graded materials (FGMs) and laminated composites.

- Improve computing efficiency by providing superior convergence in finite element and numerical solutions.

HSDTs are particularly significant in aerospace, automotive, and civil engineering applications, where precise stress distribution and deformation analysis are essential for structural integrity and optimization [8], [9].

2. Role of f(z) Function in Shear Deformation Theories

The function f(z) represents the distribution of transverse shear stresses throughout the thickness of the beam or plate. In contrast to FSDT, which presumes a uniform distribution of shear strain, HSDTs use f(z) to guarantee that transverse shear stress adheres to traction-free boundary conditions without necessitating a correction factor [10]. The function f(z) in shear deformation theories may be succinctly described as follows [11], [12], [13]:

- Distribution of Shear Stress: The use of a suitable f(z) function enables an accurate forecast of shear stress distribution across the structure's thickness. This is essential for dense and stratified materials where shear deformation is pivotal.

- Elimination of Shear Correction Factors: In contrast to FSDT, which incorporates an empirical correction factor to address shear stress non-uniformity, HSDTs use a judiciously selected f(z) to inherently fulfill boundary criteria, hence enhancing precision.

- Impact on Structural Stiffness: The shear strain function f(z) directly influences the computation of stiffness matrices in finite element formulations, leading to enhanced accuracy in deflection and stress analysis predictions.

- Compatibility with Functionally Graded Materials (FGMs) and Laminates: In functionally graded and laminated composite materials, the mechanical characteristics fluctuate throughout the thickness, necessitating a flexible shear function f(z) that accommodates material discontinuities and progressive variations in attributes.

- Superior Approximations for Improved Precision: Diverse polynomial, exponential, and hyperbolic functions have been suggested for f(z) to encapsulate higher-order effects and enhance stress distributions in intricate systems.

The selection of the f(z) function is essential for precisely simulating transverse shear effects in structural analysis, especially in thick plates, sandwich panels, and advanced composite systems [14], [15].

This study serves as a concise overview concentrating on the function and formulation of diverse shear deformation shape functions f(z) within higher-order shear deformation theories applied to functionally graded material (FGM) plates and beams. The aim is to conduct a qualitative assessment of several f(z) functions regarding their physical consistency, mathematical framework, and consequences for finite element modeling. This review is limited to static bending issues, excluding vibration, buckling, and dynamic studies. This study seeks to summarize current advancements, elucidate theoretical benefits and constraints, and provide constructive insights for researchers using shear deformation theories in computational mechanics.

3. Commonly Used Shear Strain Distribution Functions f(z)

The selection of an appropriate shear strain distribution function f(z) is crucial in shear deformation theories (SDTs) as it directly influences the accuracy of shear stress prediction in beams, plates, and shells. Various f(z) functions have been proposed to capture shear effects accurately, particularly in thick and laminated structures [15]. Below is a comprehensive classification of commonly used and f(z) functions in structural analysis.

3.1. Polynomial-Based Functions

Polynomial functions are widely used due to their mathematical simplicity and ability to approximate smooth distributions.

3.1.1. Parabolic Functions

$$\begin{aligned} f(z) &= 1-(z/h)^2 \\ f(z) &= 1-(z/h)^4 \\ f(z) &= 1-(z/h)^6 \end{aligned} \tag{1}$$

The distribution chart of parabolic functions according to the thickness of the structure is shown in Figure 1.

Application: Used in higher-order shear deformation theories (HSDTs) to approximate shear stress variation in thick plates and functionally graded materials (FGMs).

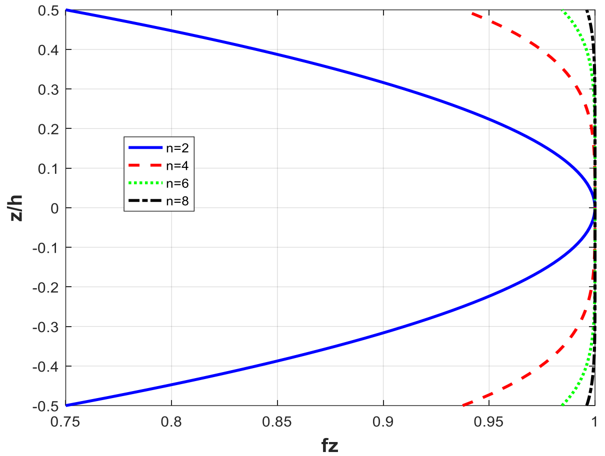

Figure 1. *Distribution of f(z) according to the parabolic function*

3.1.2. Cubic and Higher-Order Polynomials

$$\begin{aligned} f(z) &= 1-\alpha(z/h)^2+\beta(z/h)^4 \\ f(z) &= 1-\alpha(z/h)^2+\beta(z/h)^4-\gamma(z/h)^6 \end{aligned} \tag{2}$$

The distribution chart of cubic and higher-order polynomials according to the thickness of the structure is shown in Figure 2.

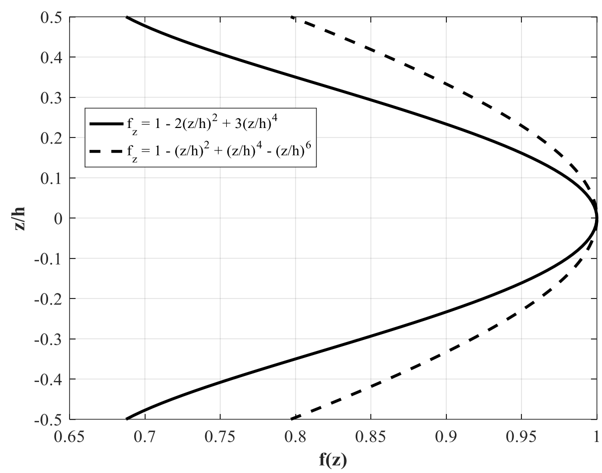

Figure 2. *The distribution chart of cubic and higher-order polynomials*

Application: Used in improved HSDTs for better accuracy in laminated composites and sandwich structures.

3.2. Trigonometric Functions

Trigonometric shear strain functions offer a smoother and more accurate representation of the shear strain distribution compared to polynomials.

3.2.1. Sine and Cosine Functions

$$f(z)=sin\left(\frac{\pi z}{2h}\right); f(z)=cos\left(\frac{\pi z}{2h}\right); f(z)=cos\left(\frac{\pi z}{h}\right) \tag{3}$$

The distribution chart of sine and cosine functions according to the thickness of the structure is shown in Figure 3.

Application: Common in refined higher-order theories, improving shear correction factor-free modeling in sandwich and composite plates.

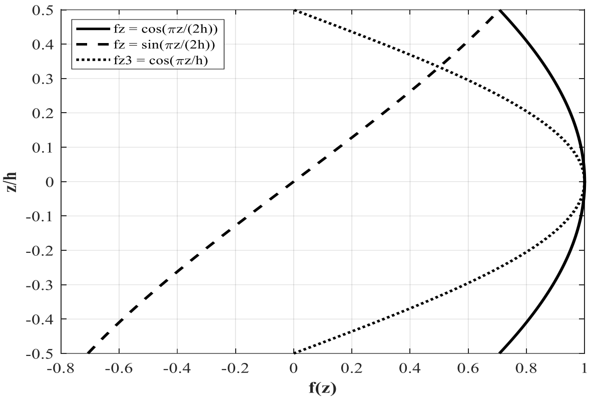

Figure 3. *Distribution of f(z) according to sine and cosine functions*

3.2.2. Hyperbolic Functions

$$f\left(z\right)=sinh\left(\frac{kz}{h}\right)$$
$$f\left(z\right)=tanh\left(\frac{kz}{h}\right) \quad (4)$$

The distribution chart of hyperbolic functions according to the thickness of the structure is shown in Figure 4.

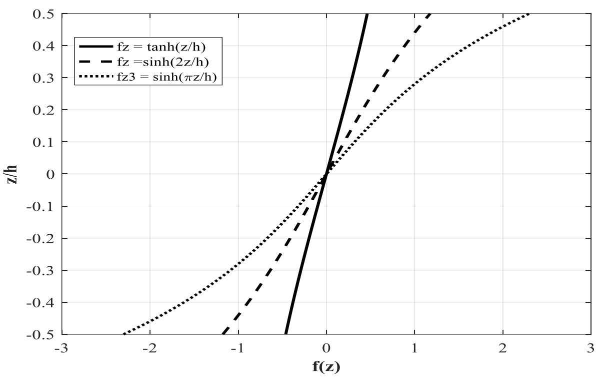

Figure 4. *Hyperbolic distribution of f(z)*

Application: Suitable for thick beams and plates, capturing rapid shear variations near the neutral axis.

3.3. Rational and Exponential Functions

These functions offer improved accuracy for highly non-uniform materials, such as functionally graded structures.

3.3.1. Rational Functions

$$f\left(z\right)=\frac{1}{1+\left(z/h\right)^{n}} \quad (5)$$

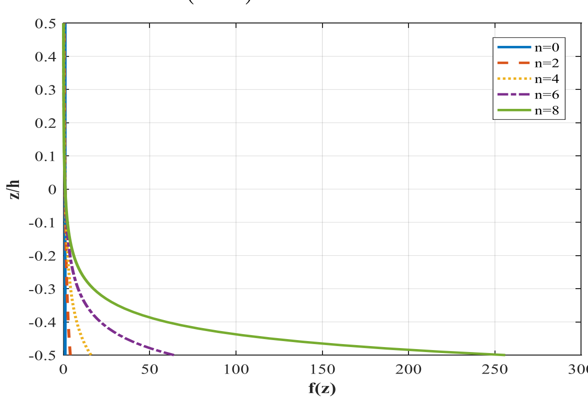

Figure 5. *Distribution of f(z) according to rational functions*

The distribution chart of rational functions according to the thickness of the structure is shown in Figure 5.

Application: Used in thick composite and sandwich plates to better approximate real shear stress distributions.

3.3.2. Exponential Functions

$$f\left(z\right)=e^{-\lambda\left(z/h\right)}$$
$$f\left(z\right)=1-e^{-\lambda\left(z/h\right)} \quad (6)$$

The distribution chart of exponential functions according to the thickness of the structure is shown in Figure 6.

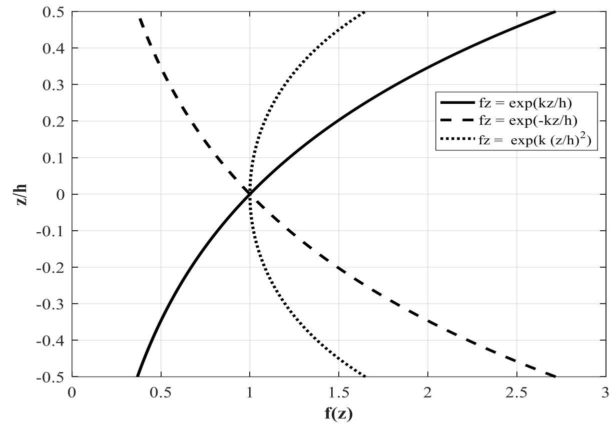

Figure 6. *Distribution of f(z) according to exponential functions*

Application: Applied to exponential-FGM plates and structures with strong material gradation.

3.4. Layer-Wise and Piecewise-Defined Functions

These functions are tailored for laminated and multi-layered composite structures where shear strain variation differs significantly between layers.

3.4.1. Layer-Wise Models

$$f_i\left(z\right)=1-\left(z_i/h_i\right)^2 \text{ for layer } i \quad (7)$$

The distribution chart of layer-wise models, categorized by the thickness of the structure, is shown in Figure 7.

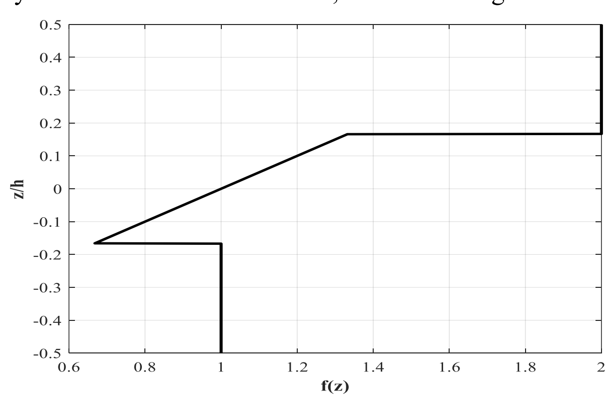

Figure 7. *Distribution of f(z) by layer and fractional form functions*

Application: Used in multi-layer plate theories, particularly in laminated composites and sandwich structures.

3.4.2. Piecewise Continuous Functions

$$f\left(z\right)=f_1\left(z\right) \text{ for layer } i \quad (8)$$

Application: Used in hybrid composite plates where different layers have distinct shear deformation properties.

In general, the asymptotically exact theory, as in Le [13] and other authors, has provided a very powerful framework for modeling the mechanical behavior of FGM plates/beams. However, this theory often requires more complex representations and heavier computations, especially when considering complex boundary conditions or modeling on finite element meshes.

The shear strain distribution function f(z) that we propose has the advantage of being simple and easy to implement in finite element models and still provides high accuracy in the application domain of interest. Although it does not reach the asymptotic accuracy level of Le [13], this function f(z) still preserves the main physical properties, such as the boundary stress condition, without correction.

Table 1. *Overview of prevalent shear deformation theories*

Theory	Shear strain shape function f(z)	Accuracy	Correction factor
First-order Shear Deformation Theory (FSDT / Mindlin)	Constant (linear distribution of shear strain)	Low–moderate (depends on thickness)	Yes
Reddy's Third-order Theory (HSDT)	Parabolic	High	No
Touratier [16]	Trigonometric	Very high	No
Thai and Kim [17], [18]	Polynomial	Very high	No
Karama et al.	Hyperbolic tangent	High	No
Carrera Unified Formulation (CUF)	Generalized f(z) via Taylor/ Legendre expansion	Very high (adjustable order)	No
Le – AET [13]	Derived from an asymptotically exact solution	Very high (reference-level)	No

4. Evaluation of the suggested synthetic shear deformation function quantitatively

In this part, a benchmark problem provides a quantitative example of the suggested approach.

4.1. Problem description

Analysis is done on a square functionally graded material (FGM) plate that is simply supported, has side length a, thickness h, and is subjected to a uniformly distributed transverse load q_0. A power-law distribution describes how the material's characteristics change with thickness:

$$P(z) = P_m + (P_c - P_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n \tag{9}$$

where $n = 2$ is the material gradient index. The geometric ratio is taken as $a/h = 20$. The dimensionless deflection is defined as:

$$\bar{w} = \frac{w_{max} D}{q_0 a^4} \tag{10}$$

where D is the flexural rigidity.

4.2. Proposed synthesized shear deformation function

In this work, the displacement field at an arbitrary point of the FG plate, denoted by (u_x, u_y, u_z), is presented as follows [19]:

$$\bar{\boldsymbol{u}} = \boldsymbol{u}_1 + z\boldsymbol{u}_2 + f(z)\boldsymbol{u}_3 \tag{11}$$

where

$$\bar{\boldsymbol{u}} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}; \boldsymbol{u}_1 = \begin{Bmatrix} u_0 \\ u_0 \\ u_0 \end{Bmatrix}; \boldsymbol{u}_2 = \begin{Bmatrix} -\dfrac{\partial w_0}{\partial x} \\ -\dfrac{\partial w_0}{\partial y} \\ 0 \end{Bmatrix}; \boldsymbol{u}_3 = \begin{Bmatrix} \varphi_x \\ \varphi_y \\ 0 \end{Bmatrix} \tag{12}$$

A weighted collection of representative functions - such as cubic, sinusoidal, and hyperbolic forms - that are frequently employed in higher-order shear deformation theories is used to generate the transverse shear deformation function:

in which a_1, a_2, a_3 are weighting coefficients.

$$f(z) = a_1\left(z - \frac{4z^3}{3h^2}\right) + a_2\frac{h}{\pi}sin\left(\frac{\pi z}{h}\right) + a_3 h\,tanh\left(\frac{z}{h}\right) \tag{13}$$

The coefficients are established by a minimization process aimed at minimizing the deviation from a reference solution (Reddy's third-order theory). The optimum values in this study are $a_1 = 0.28$, $a_2 = 0.47$, $a_3 = 0.25$.

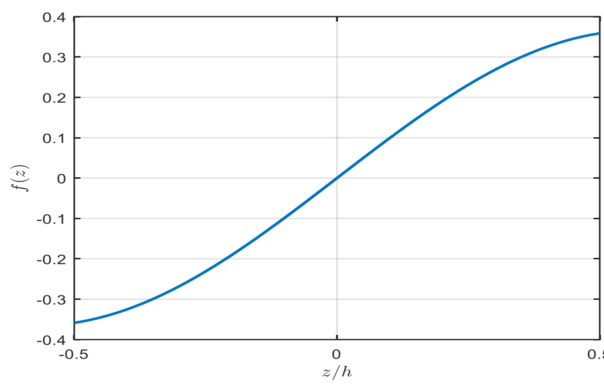

Figure 8. *Thickness-wise distribution of the proposed shear function*

This formulation allows the model to adopt the traits of many shear functions while preserving flexibility and precision.

The current findings are juxtaposed with other accepted shear deformation hypotheses documented in the literature, including the following: Reddy (third-order shear deformation theory), Zenkour (analytical functionally graded material plate solution), Thai and Kim (refined plate theory).

Table 2. *Comparison of dimensionless maximum deflection*

Model	**Theory**	$\bar{w}_{\max}$	**Deviation (%)**
Reddy [20]	TSDT	1.000	0.00
Zenkour [21]	Analytical	0.988	1.20
Thai & Kim [22]	Refined	1.006	0.60
Present (proposed)	Synthesized $f(z)$	0.998	0.20

Table 3. *Comparison of transverse shear stress*

Model	$\overline{\tau}_{zx}^{\max}$	Deviation (%)
Reddy [20]	1.000	0.00
Zenkour [21]	0.965	3.50
Thai & Kim [22]	1.015	1.50
Present	0.995	0.50

The results in Tables 2 and 3 indicate that the suggested synthesized shear deformation function aligns closely with established theories. The divergence from Reddy's third-order shear deformation theory, considered a benchmark solution, is merely 0.2%. In comparison to other sophisticated models, including those developed by Zenkour, Thai, and Kim, the current formulation demonstrates equivalent or enhanced accuracy. The synthesized function accurately encapsulates transverse shear effects without necessitating a predetermined analytical form. It has been noted that various classical shear deformation functions may either understate or overstate the structural response. In contrast, the suggested weighted formulation reconciles these inclinations, resulting in enhanced predictive performance. This outcome validates that the suggested framework is both precise and adaptable, rendering it appropriate for the analysis of intricate FGM systems. This comparison functions as a basic quantitative illustration, as indicated by the reviewer. A more extensive parametric analysis will be performed in further research.

5. Conclusion

The development of shear deformation theories (SDTs) represents significant progress in structural analysis, enabling more accurate and efficient simulation of shear effects in many engineering applications. Traditional first-order shear deformation theories (FSDTs) and classical plate theories (CPTs) often fail to accurately depict the complex shear strain distribution, especially in thick, composite, and functionally graded materials (FGMs). Higher-order shear deformation theories (HSDTs), including revised shear strain distribution functions f(z), have markedly improved the accuracy of stress and displacement predictions, eliminating the need for empirical shear correction factors.

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